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Running electromagnetic coupling constant: low energy normalization and the value at M_Z

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Abstract

A numerical value for the running electromagnetic coupling constant in the $\overline{\text{MS}}$ scheme is calculated at the low energy normalization scale equal to the τ-lepton mass M_τ. This low energy boundary value is used for running the electromagnetic coupling constant to larger scales where high precision experimental measurements can be performed. Particular scales of interest are the b-quark mass for studying the Υ-resonance physics and the Z-boson mass M_Z for high precision tests of the standard model and for the Higgs mass determination from radiative corrections. A numerical value for the running electromagnetic coupling constant at M_Z in the on-shell renormalization scheme is also given.

1   Introduction.

Dimensional regularization (DR) and minimal subtraction (MS) are convenient and widely used technical tools for perturbative calculations in particle phenomenology [1, 2]. In low orders of perturbation theory (PT) dimensional regularization does not give any decisive computational advantage. However, the high order PT many-loop calculations are rather involved and, in practice, only dimensional regularization supplemented by the recurrence relations based on the integration-by-part technique [3] allowed one to obtain new analytical results, e.g. [4]. Minimal subtraction, being a simple method of renormalizing the dimensionally regularized PT diagrams, is now becoming a dominant way of theory parameterization in a form of the $\overline{\text{MS}}$ scheme [5]. The renormalization in the $\overline{\text{MS}}$ scheme is mass independent that allows an efficient computation of renormalization group functions describing scaling of $\overline{\text{MS}}$ parameters. However, the mass independence of the renormalization procedure is physically inconvenient because decoupling of heavy particles is not automatic [6]. The physical property of decoupling is restored within an effective theory approach with the explicit separation of different mass scales such that the parameters of neighboring effective theories (couplings, masses, ...) should be sewed (matched) near the point where a new scale appears. This machinery, worked out up to three-loop order in PT, allows one to compare theoretical results in the $\overline{\text{MS}}$ scheme for a variety of scales with a uniform control over the precision of PT calculation. In particular, this technique allows one to compare theoretical quantities extracted from the low energy data with results of the Z-boson peak analyses within the standard model (SM) of particle interactions. The high precision tests of SM at the Z-boson peak, completed at one-loop level, have shown a good agreement with theoretical results obtained from the low energy data. For the search of new physics and further tests of SM at the next level of precision the computations for many observables at the Z-boson peak should be done

with two-loop accuracy which presently is an actual calculational task. Because of the computational advantage of dimensional regularization in many-loop calculations, the high order PT results for theoretical amplitudes at the Z-boson peak tend to be obtained in terms of the $\overline{\text{MS}}$ scheme parameters which are natural quantities for the minimally-subtracted dimensionally-regularized diagrams. It was found that the use of the running electromagnetic (EM) coupling normalized at M_Z in the $\overline{\text{MS}}$ scheme makes PT expansions near the Z-boson peak reliable and corrections small. However, contrary to the fine structure constant α, the running EM coupling in the $\overline{\text{MS}}$ scheme has no immediate physical meaning and its numerical value is not well known. At the same time, QED, being an old part of the SM, is well tested at low energies where the fine structure constant α is a natural interaction parameter defined in a physical manner by subtraction on the photon mass shell. The fine structure constant is very well known numerically that would make it a natural reference parameter for high precision tests of SM. However, because of a huge numerical difference between the values of the photon and Z-boson masses the use of the fine structure constant for calculations at the Z-boson peak generates large corrections in PT. For applications to high precision tests of the standard model with observables near the Z-boson peak [7], one should transform α into a proper high energy parameter, i.e. into the electromagnetic coupling constant at a scale of the Z-boson mass M_Z (see e.g. [8, 9]). Then the large PT corrections are hidden (renormalized) into a numerical value of this new parameter which is more natural for describing the Z-boson peak observables than α. Therefore, a numerical value of the running EM coupling constant at M_Z is a new important number which has been chosen as a standard reference parameter [10]. A difference of the numerical value for this parameter from $\alpha^{-1} = 137.036\ldots$ should be theoretically calculated. The change is accounted for through the renormalization group (RG) technique [11, 12, 13]. Because the fine structure constant is defined at

vanishing momentum it is an infrared sensitive quantity and a contribution of strong interactions into its RG evolution cannot be computed perturbatively: the infrared region is a domain of strong coupling that requires a nonperturbative (nonPT) treatment. The contribution of the infrared (IR) region is usually taken into account within a semi-phenomenological approximation through a dispersion relation with direct integration of low energy data. There has been a renewal of interest in a precise determination of the hadronic contribution into the electromagnetic coupling constant at M_Z during the last years in connection with the constraints on the Higgs boson mass from radiative corrections in SM [14]. Some recent references giving a state-of-the-art analysis of this contribution are [15, 16, 17, 18]. A quasi-analytical approach was used in ref. [19] where some references to earlier papers can be found (see also [20, 21]). An extremely thorough data-based analysis is given in ref. [22]. However, the virtual lack of data for energies higher than $15 \div 20$ GeV makes it unavoidable to use theoretical formulas in the dispersion relation. Fortunately, the theoretical results necessary for electromagnetic current correlators (the photon vacuum polarization function) are known in high orders of PT and are reliable at large energies. Therefore, the real value of dispersion relations is to find a boundary condition for the running EM coupling at a low energy normalization scale where data are accurate. If this low energy normalization scale is large enough for strong interaction PT to be applicable then the renormalization group can be used to run the initial value to any larger scale with very high precision. The running of the electromagnetic coupling constant can be defined in different ways depending on the renormalization procedure chosen. The evolution can be described in both on-shell and $\overline{\text{MS}}$ schemes: the corresponding β-functions are available with high precision. The recent calculation of the numerical value for the running EM coupling at M_Z with evolution in the $\overline{\text{MS}}$ scheme has been presented in ref. [23].

In the present paper I calculate a low energy boundary condition for the running electromagnetic coupling constant in the $\overline{\text{MS}}$ scheme using almost no experimental data but masses of ground states in the ρ- and φ-meson channels. The necessary IR modification of the light quark spectrum is determined by consistency with OPE. The theoretical parameters of the calculation are the strong coupling constant $\alpha_s(M_\tau)$, the strange quark mass $m_s(M_\tau)$ and the gluon and quark vacuum condensates. The numerical values for these parameters accumulate a lot of information on low energy data presented in the standard rate $R(s)$ of e^+e^- annihilation into hadrons. Therefore, the present calculation compresses low energy data into the numerical values of several key theoretical parameters that allows one to perform an analysis of the IR domain necessary for determination of the low energy boundary value for the running EM coupling. The evolution to larger scales is straightforward and very precise within perturbation theory.

2   Basic relations

The relation between the running EM coupling constant $\bar{\alpha}(\mu)$ in the $\overline{\text{MS}}$ scheme and the fine structure constant α is obtained by considering the photon vacuum polarization function. The correlator of the EM current j_μ^{em}

$$12\pi^2 i \int \langle T j_\mu^{em}(x) j_\nu^{em}(0)\rangle e^{iqx} dx = (q_\mu q_\nu - g_{\mu\nu}q^2)\Pi_\#(q^2) \tag{1}$$

is defined with a generic scalar function $\Pi_\#(q^2)$. The particular scalar functions $\Pi(\mu^2, q^2)$ and $\Pi_{os}(q^2)$ are defined through the correlator of electromagnetic currents eq. (1) (and the generic function $\Pi_\#(q^2)$) but with different subtraction procedures to remove infinities. The first function $\Pi(\mu^2, q^2)$ is renormalized in the $\overline{\text{MS}}$ scheme and the second function $\Pi_{os}(q^2)$ is renormalized by subtraction on the photon mass-shell $q^2 = 0$ which implies the normalization condition $\Pi_{os}(0) = 0$. Note that for the actual

calculation of $\Pi_{os}(q^2)$ one can use dimensional regularization and the $\overline{\text{MS}}$ scheme in cases when $\Pi(\mu^2, 0)$ exists,

$$\Pi_{os}(q^2) = \Pi(\mu^2, q^2) - \Pi(\mu^2, 0) \,.$$

The relation between couplings and polarization functions in different schemes reads

$$\frac{3\pi}{\bar{\alpha}(\mu^2)} + \Pi(\mu^2, q^2) = \frac{3\pi}{\alpha} + \Pi_{os}(q^2) \,. \tag{2}$$

In the limit $q^2 \to 0$ one finds

$$\frac{3\pi}{\bar{\alpha}(\mu^2)} + \Pi(\mu^2, 0) = \frac{3\pi}{\alpha} \,. \tag{3}$$

Eq. (2) is related to the Coulomb law for charged particles. For the potential of the EM interaction of two charged leptons one finds in the $\overline{\text{MS}}$ scheme

$$V(\mathbf{q}^2) = -\frac{4\pi\bar{\alpha}(\mu^2)}{\mathbf{q}^2} \frac{1}{1 + \frac{\bar{\alpha}(\mu^2)}{3\pi}\Pi(\mu^2, \mathbf{q}^2)} \,. \tag{4}$$

This expression is μ independent because of RG invariance. Being expressed through the fine structure constant α the Coulomb potential reads

$$V(\mathbf{q}^2) = -\frac{4\pi\alpha}{\mathbf{q}^2} \frac{1}{1 + \frac{\alpha}{3\pi}\Pi_{os}(\mathbf{q}^2)} \tag{5}$$

with $\Pi_{os}(0) = 0$. The limit of large distances

$$4\pi\alpha = -\lim_{\mathbf{q}^2 \to 0} \mathbf{q}^2 V(\mathbf{q}^2) \tag{6}$$

gives the fine structure constant. In the Coulomb law eqs. (4,5) $q = (0, \mathbf{q})$ and $q^2 = -\mathbf{q}^2$. This makes q^2 in eq. (2) essentially Euclidean. We keep notation $\mathbf{q}^2$ for a positive number to stress the calculation in the Euclidean domain. Eq. (3) is just a relation between schemes in which the EM coupling is defined or finite renormalization.

For massless quarks the limit $\mathbf{q}^2 \to 0$ in eq. (2) requires care. The polarization function $\Pi(\mu^2, 0)$ cannot be calculated in PT if strong interactions are included because light quarks are essentially massless.

Besides the $\overline{\text{MS}}$ running coupling constant $\bar{\alpha}(\mu)$, the on-shell running coupling $\alpha_{os}(\mathbf{q}^2)$, which can also be used in the Z-boson peak analyses, is defined through

$$\alpha_{os}(\mathbf{q}^2) = \frac{\alpha}{1 + \frac{\alpha}{3\pi}\Pi_{os}(\mathbf{q}^2)}, \quad \alpha_{os}(0) = \alpha. \tag{7}$$

The numerical value for the on-shell running coupling $\alpha_{os}(\mathbf{q}^2)$ can be found from eq. (2) if $\bar{\alpha}(\mu)$ is known and $\Pi(\mu^2, \mathbf{q}^2)$ is calculable for a given $\mathbf{q}^2$.

In the present paper I calculate the low energy boundary condition for the running EM coupling in the $\overline{\text{MS}}$ scheme, i.e. the value $\bar{\alpha}(\mu_0)$ at some μ_0. A convenient scale is the τ lepton mass M_τ which is large enough for strong interaction PT to work, i.e. $\mu_0 = M_\tau$. The value $\bar{\alpha}(M_\tau)$ can then be run to other scales with the standard RG equation. The particular values of interest are m_b for Υ physics and M_Z for high precision SM tests and Higgs boson search. The RG functions in the $\overline{\text{MS}}$ scheme are known with a very high accuracy that makes the running very precise numerically.

3 Low energy normalization: formulas

One needs a numerical value of the polarization function $\Pi(\mu^2, q^2)$ at $q^2 = 0$ at some low normalization point μ^2. There are lepton and quark contributions to the EM current (see a note about W bosons below). Because decoupling is not explicit we count only contributions of particles which are considered active for a given scale.

3.1 Leptons

For a lepton 'l' with the pole mass M_l we retain masses that makes $\Pi(\mu^2, 0)$ directly computable in low orders of PT where strong interactions are absent. The matching

condition reads

$$\Pi^l(\mu^2, 0) = \ln \frac{\mu^2}{M_l^2} + \frac{\bar{\alpha}(\mu^2)}{\pi} \left(\frac{45}{16} + \frac{3}{4} \ln \frac{\mu^2}{M_l^2} \right) + \mathcal{O}(\bar{\alpha}^2). \tag{8}$$

Note that $\mathcal{O}(\alpha^2)$ corrections are also available [24] but they are totally negligible numerically for our purposes. With accuracy of order α there is no numerical difference between the fine structure constant α and the running coupling constant $\bar{\alpha}(\mu^2)$ in the RHS of eq. (8). For numerical estimates we substitute α. For $\mu = M_l$ the lepton 'l' decouples completely in the leading order (which can be practical for the τ lepton). Because the fine structure constant α is small numerically we do not resum the expression in the RHS of eq. (8). Note that the expression (8) can basically be used at any μ. In a sense the matching for leptons can be done just at any scale of interest, for instance, at $\mu = M_Z$. For a lepton we use the pole mass M_l which is a meaningful parameter in finite order PT. Eq. (8) gives the leptonic part of finite renormalization between the running and fine structure constants in eq. (3). In eq. (8) we neglect strong interactions (quark contributions) which appear in $\mathcal{O}(\alpha^2)$ order. If strong interactions are included then one cannot use PT with such a low scale as the electron or muon mass and the full IR analysis analogous to that done for light quarks (see below) is necessary. Eq. (8) solves the lepton part of the normalization condition.

3.2   Light quarks

For the hadronic part of the vacuum polarization function we first consider a light quark contribution which is most complicated. For the light (massless) quarks the limit $\mathbf{q^2} \to 0$ in eq. (2) necessary to relate the running coupling to the fine structure constant cannot be performed in PT. This is, however, an infrared (IR) problem which is unsolved in QCD within PT. The low energy domain is not described in QCD with massless quarks by PT means and PT expressions should be modified for the limit $\mathbf{q^2} \to 0$ in eq. (2) to exist. Such a modification must not change an ultraviolet (UV)

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structure of the correlators because RG invariance should be respected. Therefore, it is convenient to perform an IR modification using dispersion relations which give contributions of different energy ranges separately. There are three potentially IR dangerous quarks u, d, and s. For matching light quarks we work in $n_f = 3$ effective theory, i.e. in QCD with three active light quarks.

A note about notation is in order. We consider a generic light quark correlator normalized at the parton level to 1 (as for its asymptotic spectral density). Then we add necessary factors to account for color and/or charge structure. Thus, for u quark, for instance,

$$\Pi^u(\mathbf{q}^2) = N_c e_u^2 \Pi^{light}(\mathbf{q}^2) \tag{9}$$

where $e_u = 2/3$ is a u-quark EM charge and $N_c = 3$ is a number of colors. For light quarks the PT part of the correlator is calculable for large $\mathbf{q}^2$ and reads in the $\overline{\text{MS}}$ scheme (e.g. [25])

$$\Pi^{light}(\mu^2, \mathbf{q}^2) = \ln\frac{\mu^2}{\mathbf{q}^2} + \frac{5}{3} + a_s\left(\ln\frac{\mu^2}{\mathbf{q}^2} + \frac{55}{12} - 4\zeta(3)\right)$$

$$+ a_s^2\left(\frac{9}{8}\ln^2\frac{\mu^2}{\mathbf{q}^2} + \left(\frac{299}{24} - 9\zeta(3)\right)\ln\frac{\mu^2}{\mathbf{q}^2} + \frac{34525}{864} - \frac{715}{18}\zeta(3) + \frac{25}{3}\zeta(5)\right) \tag{10}$$

where $a_s = \alpha_s/\pi$, $a_s \equiv a_s^{(3)}(\mu)$. Eq. (10) is written for $n_f = 3$ active light quarks. The limit $\mathbf{q}^2 \to 0$ cannot be performed because there is no scale for light quarks and no PT expression as eq. (8) is available.

Still small momenta mean IR problems and we want to modify only the IR structure of the correlator $\Pi^{light}(\mu^2, \mathbf{q}^2)$. It is convenient to modify just the contribution of low energy states into the correlator which can be done through a dispersion relation. The dispersion relation reads

$$\Pi^{light}(\mathbf{q}^2) = \int\limits_0^\infty \frac{\rho^{light}(s)ds}{s + \mathbf{q}^2} \tag{11}$$

where dimensional regularization is understood for $\rho^{light}(s)$. In fact, eq. (11) can be used for bare quantities $\Pi^{light}(\mathbf{q}^2)$ and $\rho^{light}(s)$. The limit $\mathbf{q}^2 \to 0$ in the RHS of

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eq. (11) is IR-singular and cannot be performed if the PT expression for the spectral density $\rho^{light}(s)$ is used. Therefore, one should modify the low energy behavior of the spectrum where PT is not applicable. If such a modification is local (has only a finite support in energy variable s in eq. (11)) then it does not affect any UV properties (μ^2 dependence) of $\Pi^{light}(\mu^2, \mathbf{q}^2)$ which are important for RG. The low energy modification is inspired by experiment: at low energies there is a well-pronounced bound state as a result of strong interaction. We, therefore, adopt a model of IR modification that the high energy tail of the integral in eq. (11) is computed in PT (duality arguments) that retains RG structure of the result while in the low energy domain there is a contribution of a single resonance. An IR modification is performed for contributions of u, d and s quarks. The massless u and d quarks interact with photons through isotopic combinations $I = 1$ (ρ-meson channel) and $I = 0$ (ω-meson channel). For our purposes these two channels are completely degenerate and are treated simultaneously. The s-quark contribution is considered separately because of its nonvanishing (small) mass m_s.

For a generic light quark correlator in the massless PT approximation we introduce the IR modification

$$\rho^{light}(s) \rightarrow \rho^{light}_{IRmod}(s) = F_R \delta(s - m_R^2) + \rho^{light}(s)\theta(s - s_0) \tag{12}$$

where F_R, m_R and s_0 are IR parameters of the spectrum. Note that they are not necessarily immediate numbers from experiment. Substituting the IR modified spectrum (12) into eq. (11) one finds

$$\Pi^{light}_{IRmod}(\mu^2, 0) = \frac{F_R}{m_R^2} + \ln\frac{\mu^2}{s_0} + \frac{5}{3} + a_s\left(\ln\frac{\mu^2}{s_0} + \frac{55}{12} - 4\zeta(3)\right)$$

$$+ a_s^2\left(\frac{9}{8}\ln^2\frac{\mu^2}{s_0} + \left(\frac{299}{24} - 9\zeta(3)\right)\ln\frac{\mu^2}{s_0} + \frac{34525}{864} - \frac{715}{18}\zeta(3) + \frac{25}{3}\zeta(5) - \frac{3\pi^2}{8}\right). \tag{13}$$

Here $a_s \equiv a_s^{(3)}(\mu)$. We identify m_R with a mass of the lowest resonance which is the

only input giving a scale to the problem. The IR modifying parameters F_R and s_0 are fixed from duality arguments.

Notice the difference in the $\mathcal{O}(a_s^2)$ order between eq. (10) and eq. (13): in eq. (13) there is a new term $-3\pi^2/8$. This is so called 'π^2' correction (e.g. [26]). It can be rewritten through $\zeta(2) = \pi^2/6$.

To describe the IR structure of the correlator in representation given by eq. (13) we use OPE with power corrections that semi-phenomenologically encode information about the low energy domain of the spectrum through the vacuum condensates of local gauge invariant operators [27]. The OPE for the light quark correlator reads

$$\Pi^{OPE}(\mu^2, \mathbf{q}^2) = \Pi^{light}(\mu^2, \mathbf{q}^2) + \frac{\langle \mathcal{O}_4 \rangle}{\mathbf{q}^4} + \mathcal{O}\left(\frac{\langle \mathcal{O}_6 \rangle}{\mathbf{q}^6}\right). \tag{14}$$

The quantities $\langle \mathcal{O}_{4,6} \rangle$ give nonPT contributions of dimension-four and dimension-six vacuum condensates. These contributions are UV soft (they do not change short distance properties) and related to the IR modification of the spectrum. For the purposes of fixing the numerical values of the parameters F_R and s_0 which describe the IR modification of the spectrum one needs only first two power corrections $1/\mathbf{q}^2$ and $1/\mathbf{q}^4$; the coefficient of the $1/\mathbf{q}^2$ correction vanishes because there are no gauge invariant dimension-two operators in the massless limit. Computing the IR modified polarization function and comparing it with the OPE expression we find finite energy sum rules (FESR) for fixing the parameters F_R and s_0 [28]. The system of sum rules has the form

$$
\begin{aligned}
F_R &= s_0 \left\{ 1 + a_s + a_s^2 (\beta_0 \ln \frac{\mu^2}{s_0} + k_1 + \beta_0) \right\} + \mathcal{O}(a_s^3), \\
F_R m_R^2 &= \frac{s_0^2}{2} \left\{ 1 + a_s + a_s^2 \left(\beta_0 \ln \frac{\mu^2}{s_0} + k_1 + \frac{\beta_0}{2} \right) \right\} - \langle \mathcal{O}_4 \rangle + \mathcal{O}(a_s^3).
\end{aligned}
\tag{15}
$$

Here $\beta_0 = 9/4$ and

$$k_1 = \frac{299}{24} - 9\zeta(3).$$

We treat $\langle \mathcal{O}_4 \rangle$ as a small correction and take its coefficient function as a constant (the total contribution is RG invariant). Eqs. (15) fix F_R and s_0 through m_R^2 and $\langle \mathcal{O}_4 \rangle$. Using higher order terms in OPE expansion ($\langle \mathcal{O}_6 \rangle / \mathbf{q}^6$) one can avoid substituting m_R^2 from experiment because within the IR modification given in eq. (12) the IR scale is determined by the dimension-six vacuum condensate $\langle \mathcal{O}_6 \rangle$ [28]. We do not do that because the primary purpose is to find the normalization for the EM coupling and not to describe the spectrum in the low energy domain. The use of the experimental value for the resonance mass m_R^2 makes the calculation more precise because the numerical value for $\langle \mathcal{O}_6 \rangle$ condensate is not known well (cf ref. [29]).

The leading order solution to eqs. (15) (upon neglecting the PT and nonPT corrections) is given by the partonic model result $s_0 = 2m_R^2$, $F_R = s_0 = 2m_R^2$, which is rather precise. This solution has been used for predicting masses and residues of the radial excitations of vector mesons within the local duality approach when the experimental spectrum is approximated by a sequence of infinitely narrow resonances [30]. Such an approximation for the experimental spectrum is justified by theoretical considerations in the large N_c limit [31] and by the exact solution for two dimensional QCD [32]. For the experimental spectrum of infinitely narrow resonances the local duality approach means averaging over the energy interval around a single resonance [30]. It is expected to be less precise than the global duality method in which the average is performed over the entire spectrum. However, within the global duality approach only the total contribution of all hadronic states can be studied while the local duality can be used even for first resonances and can predict characteristics of individual hadronic states.

An accurate treatment of eqs. (15) gives the solution

$$
\begin{aligned}
s_0 &= 2m_R^2 \left(1 + \frac{\beta_0}{2} a_s^2 \right) + \frac{\langle \mathcal{O}_4 \rangle}{m_R^2} (1 - a_s) \,, \\
\frac{F_R}{m_R^2} &= 2 \left\{ 1 + a_s + a_s^2 \left(\beta_0 \ln \frac{\mu^2}{2m_R^2} + k_1 + \frac{3}{2} \beta_0 \right) \right\} + \frac{\langle \mathcal{O}_4 \rangle}{m_R^4} \,.
\end{aligned}
\tag{16}
$$

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In the solution given in eq. (16) only linear terms in the nonPT correction $\langle \mathcal{O}_4 \rangle$ are retained. This is well justified numerically. The $a_s^2 \langle \mathcal{O}_4 \rangle$ terms are neglected because the coefficient function of the $\langle \mathcal{O}_4 \rangle$ condensate is not known with such precision. In eq. (13) the scale parameter is s_0 while we solve the system (16) in terms of m_R that we identify with the resonance mass and take from experiment. Therefore, we express the PT scale s_0 through m_R according to the solution given in eqs. (16). The expansion of the log-term in eq. (13) reads

$$\ln \frac{\mu^2}{s_0} = \ln \frac{\mu^2}{2m_R^2} - \frac{\beta_0}{2} a_s^2 - \frac{\langle \mathcal{O}_4 \rangle}{2m_R^4}(1 - a_s).$$

With these results one finds an expression for the IR modified polarization function of light quarks at the origin

$$\Pi_{IRmod}^{light}(\mu^2, 0) = 2 \left\{ 1 + a_s + a_s^2 \left(\beta_0 \ln \frac{\mu^2}{2m_R^2} + k_1 + \frac{3}{2}\beta_0 \right) \right\} + \frac{\langle \mathcal{O}_4 \rangle}{m_R^4}$$

$$+ \ln \frac{\mu^2}{2m_R^2} - \frac{\beta_0}{2} a_s^2 - \frac{\langle \mathcal{O}_4 \rangle}{2m_R^4}(1 - a_s) + \frac{5}{3} + a_s \left(\ln \frac{\mu^2}{2m_R^2} - \frac{\langle \mathcal{O}_4 \rangle}{2m_R^4} + \frac{55}{12} - 4\zeta(3) \right)$$

$$+ a_s^2 \left(\frac{9}{8} \ln^2 \frac{\mu^2}{2m_R^2} + \left(\frac{299}{24} - 9\zeta(3) \right) \ln \frac{\mu^2}{2m_R^2} + \frac{34525}{864} - \frac{715}{18}\zeta(3) + \frac{25}{3}\zeta(5) - \frac{3\pi^2}{8} \right).$$

Here the first line gives the resonance contribution while the rest is the high energy tail (continuum contribution) which is computed in PT. Finally,

$$\Pi_{IRmod}^{light}(\mu^2, 0) = 2 \left\{ 1 + a_s + a_s^2 \left(\beta_0 \ln \frac{\mu^2}{2m_R^2} + k_1 + \frac{3}{2}\beta_0 \right) \right\} + \frac{\langle \mathcal{O}_4 \rangle}{2m_R^4}$$

$$+ \ln \frac{\mu^2}{2m_R^2} - \frac{\beta_0}{2} a_s^2 + \frac{5}{3} + a_s \left(\ln \frac{\mu^2}{2m_R^2} + \frac{55}{12} - 4\zeta(3) \right)$$

$$+ a_s^2 \left(\frac{\beta_0}{2} \ln^2 \frac{\mu^2}{2m_R^2} + k_1 \ln \frac{\mu^2}{2m_R^2} + \frac{34525}{864} - \frac{715}{18}\zeta(3) + \frac{25}{3}\zeta(5) - \frac{3\pi^2}{8} \right). \tag{17}$$

Eq. (17) gives $\Pi_{IRmod}^{light}(\mu^2, 0)$ as an explicit function of the nonPT scale m_R (to be taken from experiment) and theoretical quantities a_s and $\langle \mathcal{O}_4 \rangle$. The choice of the numerical value for a_s is discussed in detail later.

The condensate of dimension-four operators for light quarks is given by

$$\langle \mathcal{O}_4 \rangle = \frac{\pi^2}{3}\left(1+\frac{7}{6}a_s\right)\langle\frac{\alpha_s}{\pi}G^2\rangle + 2\pi^2\left(1+\frac{1}{3}a_s\right)(m_u+m_d)(\langle\bar{u}u\rangle + \langle\bar{d}d\rangle)\,. \qquad (18)$$

We retain small corrections proportional to the light quark masses and treat them in the approximation of isotopic symmetry for the light quark condensates $\langle\bar{u}u\rangle = \langle\bar{d}d\rangle$ which is rather precise for u and d quarks. The quark condensate part of eq. (18) is given by the PCAC relation for the π meson

$$(m_u+m_d)\langle\bar{u}u+\bar{d}d\rangle = -f_\pi^2 m_\pi^2\,.$$

Here $f_\pi = 133$ MeV is a charged pion decay constant and $m_\pi = 139.6$ MeV is a charged pion mass. For the standard numerical value of the gluon condensate $\langle\frac{\alpha_s}{\pi}G^2\rangle = 0.012$ GeV^4 [27] and $a_s = 0.1$ one finds

$$\langle \mathcal{O}_4 \rangle = \frac{\pi^2}{3}\left(1+\frac{7}{6}a_s\right)\langle\frac{\alpha_s}{\pi}G^2\rangle - 2\pi^2\left(1+\frac{1}{3}a_s\right)f_\pi^2 m_\pi^2 = 0.037 \text{ GeV}^4\,. \qquad (19)$$

For the most important contribution of u and d quarks (the u-quark contribution is enhanced by factor 4 because of its doubled electric charge in comparison to the other light quarks) the relation $s_0 = 2m_\rho^2$, where $m_\rho = 768.5$ MeV is a mass of the lowest (ρ meson) resonance in the non-strange isotopic $I = 1$ vector channel, is rather precise numerically. The gluon condensate gives a small correction to the basic duality relation for light quarks $s_0 = 2m_R^2$. Note that we do not identify F_R with the experimental number available from the analysis of the ρ-meson leptonic width but treat it as an IR modifying parameter which should be fixed from the consistency requirement with OPE. It is close numerically to its experimental counterpart because it is known that OPE gives rather an accurate description of the physical spectrum if vacuum condensates are included. In the present paper we stick to a theoretical description of the IR domain and use the lowest resonance mass as the only input for the IR modification. The same is true for the $I = 0$ channel where the lowest

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resonance is the ω meson with a mass $m_\omega = 781.94$ MeV. We do not distinguish these two channels. We consider parameters F_R and s_0 as the IR modifiers fixed theoretically through OPE and do not attempt to substitute them from experiment (using leptonic decay widths for F_R or the shape of the spectrum for s_0).

Note that the IR parameters of the spectrum F_R, m_R and s_0 are μ independent. It can be seen explicitly from eqs. (15).

The $n_f = 3$ effective theory is valid only up to $\mathbf{q}^2 \sim m_c^2$ and, formally, there are corrections of order $\mathbf{q}^2/m_c^2$ [33]. However, in the case of current correlators these corrections are small [34, 35].

For the s quark there are also corrections due to m_s which change slightly the shape of the spectrum and the consistency equations for the IR modifiers. We consider m_s as an additional IR modifier which does not affect UV properties (renormalization in the $\overline{\text{MS}}$ scheme is mass independent) and treat it as a power correction. We write OPE for the s quark in the form

$$\Pi^{OPE,s}(\mu^2, \mathbf{q}^2) = \Pi^{light}(\mu^2, \mathbf{q}^2) - \frac{6m_s^2}{\mathbf{q}^2} + \frac{\langle \mathcal{O}_4^s \rangle}{\mathbf{q}^4} + \mathcal{O}\left(\frac{\langle \mathcal{O}_6 \rangle}{\mathbf{q}^6}\right).$$

The system of equations for fixing the parameters F_{Rs} and s_{0s} reads

$$F_{Rs} + 6m_s^2 = s_{0s}\left\{1 + a_s + a_s^2(\beta_0 \ln \frac{\mu^2}{s_{0s}} + k_1 + \beta_0)\right\} + \mathcal{O}(a_s^3),$$

$$F_s m_{Rs}^2 = \frac{s_{0s}^2}{2}\left\{1 + a_s + a_s^2\left(\beta_0 \ln \frac{\mu^2}{s_{0s}} + k_1 + \frac{\beta_0}{2}\right)\right\} - \langle \mathcal{O}_4^s \rangle + \mathcal{O}(a_s^3). \quad (20)$$

Here

$$\langle \mathcal{O}_4^s \rangle = \frac{\pi^2}{3}\left(1 + \frac{7}{6}a_s\right)\langle \frac{\alpha_s}{\pi}G^2 \rangle + 8\pi^2\left(1 + \frac{1}{3}a_s\right)m_s\langle \bar{s}s \rangle$$

is a dimension-four contribution in the strange channel. One finds the solution to eqs. (20) in the form

$$s_{0s} = 2m_{Rs}^2\left(1 + \frac{\beta_0}{2}a_s^2\right) + \frac{\langle \mathcal{O}_4^s \rangle}{m_{Rs}^2}(1 - a_s) - 6m_s^2,$$

$$\frac{F_{Rs}}{m_{Rs}^2} = 2\left\{1 + a_s + a_s^2\left(\beta_0 \ln \frac{\mu^2}{2m_{Rs}^2} + k_1 + \frac{3}{2}\beta_0\right)\right\} + \frac{\langle \mathcal{O}_4^s \rangle}{m_{Rs}^4} - 12\frac{m_s^2}{m_{Rs}^2}. \quad (21)$$

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The correction due to m_s^2 is not large. Instead of eq. (17) one has

$$\Pi_{IRmod}^{light-s}(\mu^2, 0) = \Pi_{IRmod}^{light}(\mu^2, 0) - 9\frac{m_s^2}{m_{Rs}^2} \qquad (22)$$

and $m_{Rs} = m_\varphi$ and $\langle \mathcal{O}_4^s \rangle$ should be used in the first term of eq. (22) instead of m_ρ and $\langle \mathcal{O}_4 \rangle$. Here $m_\varphi = 1019.4$ MeV is a mass of the φ-meson which is the lowest resonance in the strange channel. A numerical value for $\langle \mathcal{O}_4^s \rangle$ is obtained as follows. We use the relation (e.g. [36])

$$\frac{2m_s}{m_u + m_d} = 25.0$$

and the phenomenological result $\langle \bar{s}s \rangle = (0.8 \pm 0.2)\langle \bar{u}u \rangle$ [37] to find

$$m_s \langle \bar{s}s \rangle = 12.5 \cdot 0.8 \cdot (m_u + m_d)\langle \bar{u}u \rangle = -5.0 \cdot f_\pi^2 m_\pi^2 = -0.0017 \text{ GeV}^4 . \qquad (23)$$

One could also use the PCAC relation for the K meson

$$(m_s + m_u)\langle \bar{s}s + \bar{u}u \rangle = -f_K^2 m_K^2 + \mathcal{O}(m_s^2)$$

with $f_K = 1.17 f_\pi$ and $m_K = 493.7$ MeV. Note that the PCAC relation in the strange channel is valid only up to terms of order m_s^2 which are not completely negligible numerically compared to the pion case [38]. Therefore, we use the result given in eq. (23). For the standard value $\langle \frac{\alpha_s}{\pi} G^2 \rangle = 0.012$ GeV^4 [27] and $a_s = 0.1$ one finds

$$\langle \mathcal{O}_4^s \rangle = \frac{\pi^2}{3}\left(1 + \frac{7}{6}a_s\right)\langle \frac{\alpha_s}{\pi}G^2 \rangle + 8\pi^2\left(1 + \frac{1}{3}a_s\right)(-5.0 f_\pi^2 m_\pi^2) = -0.0965 \text{ GeV}^4 . \quad (24)$$

The correction due to $m_s \langle \bar{s}s \rangle$ is dominant in the dimension four contribution in the strange case. Because the dimension-four terms represent only small corrections to the leading results for the correlators in eqs. (17,22), the precision with which they are calculated suffices for our purposes.

For the absolute value of m_s to be substituted into m_s^2 correction we use the results of recent analyses [39] and take $m_s(M_\tau) = 130 \pm 27_{exp} \pm 9_{th}$ MeV. For $m_{Rs} = m_\varphi = 1019.4$ MeV one finds

$$\frac{m_s^2}{m_\varphi^2} = 0.0163$$

which is a small expansion parameter that justifies the treatment of m_s^2 contribution as a correction.

Note that there are attempts to use constituent masses for the light quarks and to estimate the polarization functions in the way it is done for leptons or heavy quarks. Besides being *ad hoc* (and not supported by experiment) this IR modification of the light quark correlators contradicts OPE and/or the local duality over the energy interval from the origin to $1 \div 2$ GeV.

Thus, eqs. (13,17) represent a semi-phenomenological subtraction for a light quark correlator at $\mathbf{q}^2 = 0$ based on the IR modification of the spectrum consistent with OPE. Some mismatch with OPE in orders higher than $\mathcal{O}(1/\mathbf{q}^4)$ which is possible because of simplicity of the IR modification is neglected. It is justified because we need only integral characteristics of the spectrum for calculating $\Pi_{IRmod}^{light}(\mu^2, 0)$ and are not interested in the point-wise behavior of the spectral function $\rho_{IRmod}^{light}(s)$ which is used as an auxiliary quantity in this particular instance.

3.3 Heavy quarks

Matching heavy quarks is straightforward and is similar to that of leptons. It is done within PT. For a heavy quark 'q' with the pole mass $m_q \gg \Lambda_{\text{QCD}}$ one has

$$\Pi^q(\mu^2, 0) = N_c e_q^2 \Pi^{heavy}(\mu^2, 0)$$

where $\Pi^{heavy}(\mu^2, 0)$ is a generic contribution of a heavy quark to the polarization function [40]

$$\Pi^{heavy}(\mu^2, 0) = \ln \frac{\mu^2}{m_q^2} + e_q^2 \frac{\alpha}{\pi} \left(\frac{45}{16} + \frac{3}{4} \ln \frac{\mu^2}{m_q^2} \right) + a_s \left(\frac{15}{4} + \ln \frac{\mu^2}{m_q^2} \right)$$

$$+ a_s^2 \left(\frac{41219}{2592} - \frac{917}{1296} n_l + \left(4 + \frac{4}{3} \ln 2 - \frac{2}{3} n_l \right) \zeta(2) + \frac{607}{144} \zeta(3) \right.$$

$$\left. + \left(\frac{437}{36} - \frac{7}{9} n_l \right) \ln \frac{\mu^2}{m_q^2} + \left(\frac{31}{24} - \frac{1}{12} n_l \right) \ln^2 \frac{\mu^2}{m_q^2} \right) \right\} + \mathcal{O}(\alpha^2, \alpha_s^3). \quad (25)$$

Here n_l is the number of quarks that are lighter than a heavy one, $a_s = \alpha_s/\pi$ is the strong coupling constant in the effective theory with $n_l + 1$ active quarks normalized at the scale μ. Numbers in eq. (25) are given for the pole mass of a heavy quark. We neglect the (known) EM contribution of order α^2 because it is smaller than the unknown term of order α_s^3. Eq. (25) gives a contribution of the corrected partonic model, i.e. that with a heavy quark loop in the leading approximation. There is also a contribution of heavy quark loops to the light quark vacuum polarization function that should be taken into account in constructing the effective theory with a decoupled heavy quark. This contribution is small. It reads [41]

$$\Pi^{lightheavy}(\mu^2, 0) = a_s^2 N_c \left(\sum_{i=1}^{n_l} e_i^2 \right) \left(\frac{295}{1296} - \frac{11}{72} \ln \frac{\mu^2}{m_q^2} - \frac{1}{12} \ln^2 \frac{\mu^2}{m_q^2} \right). \qquad (26)$$

Eqs. (25,26) are used for c and b quarks. Note that these formulas cannot be used for s quark. Indeed, because of α_s corrections the PT scale in eq. (25) is effectively equal to m_q and is too small for PT to be applicable in the case of the strange quark since $m_s \sim \Lambda_{QCD}$.

4 Low energy normalization: numerics

In previous sections the necessary contributions due to fermions have been written down. We are not going to consider scales larger than M_Z therefore bosonic contributions into the EM current and polarization function (namely, W boson loops) are not taken into account. The above equations describe an effective theory without W bosons which decouple at energies smaller than M_Z and should be added separately for the Z-boson peak tests.

A numerical value of the strong coupling at low energies is rather important for the whole analysis. The estimates of the strong coupling numerical value at low scales are usually based on the τ lepton decay data. Within a contour resummation technique

[42, 43] the value obtained is $\alpha_s^{(3)}(M_\tau^2) = 0.343 \pm 0.009_{exp}$. Within a RG invariant treatment of ref. [44] a slightly different value $\alpha_s^{(3)}(M_\tau^2) = 0.318 \pm 0.006_{exp} \pm 0.016_{th}$ has recently been found. The uncertainty is due to the experimental error and due to truncation of the series which is estimated within an optimistic scenario that higher order terms are still perturbative (no explicit asymptotic growth). Note that even for the optimistic scenario with a reduced theoretical error as compared to the conservative estimates, the theoretical error dominates the total uncertainty of the coupling. Contour improved results include a special resummation procedure for treating contributions generated by the running which does not necessarily improve results but definitely changes them in comparison with the finite order estimates at the present level of precision. The change is still within the error bars that makes two procedures consistent. We use the value $\alpha_s^{(3)}(M_\tau^2) = 0.318 \pm 0.017$ as our basic input for the low energy strong coupling. The central value $\alpha_s^{(3)}(M_\tau^2) = 0.318$ corresponds to $\alpha_s^{(5)}(M_Z) = 0.118$ when it is run with a four-loop β-function and three-loop matching at m_c and m_b thresholds.

Now one has everything for numerical analysis. First the value of the running EM coupling constant computed in $n_f = 4$ effective theory at $\mu = M_\tau$ that is a convenient normalization point at the physical mass scale is given. Note that the c-quark pole mass is rather close to this value. In fact, the recent estimate is $m_c = 1.8 \pm 0.2$ GeV and we take $m_c = M_\tau = 1.777$ GeV as a central value, i.e. $m_c = M_\tau \pm 0.2$ GeV. Thus, the low energy normalization value $\bar{\alpha}^{(4)}(M_\tau)$ is computed with three active leptons and four active quarks.

For the lepton contribution we use the lepton masses $M_e = 0.5110$ MeV, $M_\mu = 105.66$ MeV, $M_\tau = 1777$ MeV [10]. These values are extremely precise therefore we use them as exact and assign no errors to them. We neglect the difference between the running EM coupling $\bar{\alpha}$ and fine structure constant α in corrections (which results

in $\mathcal{O}(\alpha^2)$ shift that is numerically negligible). We use $\alpha^{-1} = 137.036$. According to eq. (8) leptons give

$$\Delta^{lept}(M_\tau^2) = \sum_{l=e,\mu,\tau} \Pi^l(M_\tau^2, 0) = \left(1 + \frac{3}{4}\frac{\alpha}{\pi}\right)\left(\ln\frac{M_\tau^2}{M_e^2} + \ln\frac{M_\tau^2}{M_\mu^2} + \ln\frac{M_\tau^2}{M_\tau^2}\right) + \frac{135}{16}\frac{\alpha}{\pi}$$

$$= 21.953 + 0.058 = 22.011 \tag{27}$$

where the first number is obtained in the limit $\alpha = 0$. The α correction is almost negligible for the normalization at the scale M_τ. Note that τ lepton gives no logarithmic contribution at the scale $\mu = M_\tau$.

The $\mathcal{O}(\bar{\alpha}^2)$ correction for the lepton contribution in the $\overline{\text{MS}}$ scheme is also available [24]. This correction is parametrically small and there are no unexpectedly large numerical coefficients (in fact, they are also small) that makes the parametric estimate based on the counting of powers of α rather precise. The sum of contributions of three leptons in $\mathcal{O}(\bar{\alpha}^2)$ order is completely negligible and we treat the leptonic contribution in eqs. (8,27) as exact.

Light quarks. From eq. (17) with $m_R = m_\rho$, and $m_{Rs} = m_\varphi$ one finds for the total light quark contribution $\Delta^{uds}(M_\tau^2)$

$$\Delta^{uds}(M_\tau^2) = \Delta^u(M_\tau^2) + \Delta^d(M_\tau^2) + \Delta^s(M_\tau^2) = \Delta^\rho(M_\tau^2) + \Delta^\omega(M_\tau^2) + \Delta^\varphi(M_\tau^2)$$

$$= \frac{4}{3}\Delta^{light}(M_\tau^2) + \frac{1}{3}\Delta^{light}(M_\tau^2) + \frac{1}{3}\Delta^{light-s}(M_\tau^2) = \frac{5}{3}\Delta^{light}(M_\tau^2) + \frac{1}{3}\Delta^{light-s}(M_\tau^2)$$

$$= 9.13662 + 5.32853 a_s + 24.9086 a_s^2$$

$$= 9.13662 + 0.53937 + 0.255214 = 9.9312\,.$$

Because the calculation is explicit we can give the previous result in more detail showing all different contributions

$$\Delta^{uds}(M_\tau^2) = 9.11165 + 0.539367\left(\frac{a_s}{0.101}\right) + 0.2552\left(\frac{a_s}{0.101}\right)^2$$

$$+ 0.08865\left(\frac{\langle\mathcal{O}_4\rangle}{0.037\text{ GeV}^4}\right) - 0.0488\left(\frac{m_s}{130\text{ MeV}}\right)^2 + 0.0149\left(\frac{\langle\mathcal{O}_4^s\rangle}{0.0965\text{ GeV}^4}\right). \tag{28}$$

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The IR part of the spectrum (resonances) and the partonic quark approximation give a dominant contribution. The QCD perturbative corrections and power corrections due to m_s and $\langle \mathcal{O}_4^\# \rangle$ condensates are small. The error is

$$\delta\Delta^{uds}(M_\tau^2) = 10.5\delta a_s + 0.09\frac{\delta\langle\mathcal{O}_4\rangle}{\langle\mathcal{O}_4\rangle} - 0.1\frac{\delta m_s}{m_s} - 0.015\frac{\delta\langle\mathcal{O}_4^s\rangle}{\langle\mathcal{O}_4^s\rangle}. \tag{29}$$

Variations $\delta\langle\mathcal{O}_4^s\rangle$ and $\delta\langle\mathcal{O}_4\rangle$ are not completely independent – both quantities contain a variation of the gluon condensate. Also the error of a_s and that of the gluon condensate are correlated (see, for instance, [45]). For estimating the total error of $\Delta^{uds}(M_\tau^2)$ through less correlated quantities one could rewrite power corrections in eq. (28) in the basis of the gluon and strange quark condensates [46]. Because the correlation is not well established numerically we neglect this effect. We consider the errors of the strong coupling a_s, of the gluon condensate for $\delta\langle\mathcal{O}_4\rangle$, of the strange quark mass m_s, and of the strange quark condensate $\langle\bar{s}s\rangle$ for $\delta\langle\mathcal{O}_4^s\rangle$ as independent and use $\delta a_s = 0.017/\pi = 0.0054$, $\delta\langle\mathcal{O}_4\rangle/\langle\mathcal{O}_4\rangle = 1/2$ due to the gluon condensate, $\delta m_s/m_s = 0.28$, $\delta\langle\mathcal{O}_4^s\rangle/\langle\mathcal{O}_4^s\rangle = 1/4$ due to $\langle\bar{s}s\rangle$. With these (conservative) estimates of uncertainties one finds

$$\delta\Delta^{uds}(M_\tau^2) = \pm0.057|_{a_s} \pm 0.045|_{\langle\mathcal{O}_4\rangle} \pm 0.028|_{m_s} \pm 0.004|_{\langle\mathcal{O}_4^s\rangle}.$$

The dominant error is due to δa_s. The gluon condensate gives a sizable error because it is enhanced by the charge structure of light (mainly u) quarks and because its uncertainty is taken to be very conservative to compensate for the possible correlation with a_s. The strange channel is suppressed by factor $1/3$ in the total sum of light quark contributions and its specific features only slightly affect the result: in the rest it is quite degenerate with u and d channels. The total error for the light quark contributions added in quadrature reads

$$\delta\Delta^{uds}(M_\tau^2) = \pm0.078.$$

The final result for the contribution of light quarks into the low energy normalization of the running EM coupling is

$$\Delta^{uds}(M_\tau^2) = 9.9312 \pm 0.078 \,. \tag{30}$$

We retain some additional digits at intermediate stages just for computational purposes.

For the c quark we use eqs. (25,26). The strong coupling constant in $n_f = 4$ effective theory is found by matching between strong coupling constant in $n_f = 3$ and $n_f = 4$ effective theories.

Matching at the pole mass scale m_P for the strong coupling has the form [47]

$$a_s^{(n_l)}(m_P^2) = a_s^{(n_l+1)}(m_P^2) \left(1 + C_2 a_s^{(n_l+1)}(m_P^2)^2 + C_3 a_s^{(n_l+1)}(m_P^2)^3 + \mathcal{O}(a_s^4)\right) \tag{31}$$

where

$$C_2 = -\frac{7}{24} \,, \tag{32}$$

$$C_3 = -\frac{80507}{27648}\zeta(3) - \frac{2}{9}\zeta(2)(\ln 2 + 3) - \frac{58933}{124416} + \frac{n_l}{9}\left(\zeta(2) + \frac{2479}{3456}\right) \,. \tag{33}$$

We solve (inverse) eq. (31) perturbatively and find the expression

$$a_s^{(n_l+1)}(m_P^2) = a_s^{(n_l)}(m_P^2) \left\{1 - C_2 a_s^{(n_l)}(m_P^2)^2 - C_3 a_s^{(n_l)}(m_P^2)^3\right\} \tag{34}$$

which is used for determination of the couplings in neighboring effective theories at their boundary scale that is chosen to be the pole mass of the heavy quark. Matching at $m_c = M_\tau = 1.777$ GeV (we remind the reader that the numerical value of the c-quark mass is chosen to be $m_c = M_\tau \pm 0.2$ GeV) with $\alpha_s^{(3)}(M_\tau^2) = 0.318$ gives $a_s^{(4)}(m_c^2 = M_\tau^2) = 0.102$ or $\alpha_s^{(4)}(m_c^2 = M_\tau^2) = 0.320$. This value for the strong coupling is used in eq. (25) for the calculation of the c-quark contribution to the finite renormalization of the EM coupling. Note that though one computes with $a_s^{(4)}(M_\tau^2)$ it can well be identified numerically with $a_s^{(3)}(M_\tau^2)$: the change due to matching is tiny and is much smaller than the error of $a_s^{(3)}(M_\tau^2)$.

According to eqs. (25,26) we have

$$\Delta^c(M_\tau^2) = \Pi^c(\mu^2 = M_\tau^2, 0)$$

$$= 0.00387 + 0.00474 + 0.51001 + 0.32817 = 0.84679 \qquad (35)$$

where the first term is EM contribution, the second one is loop contribution (eq. (26)) and the last two terms give PT expansion of direct contribution (eq. (25)). One sees that the EM and loop contributions are much smaller than the direct contribution. Convergence of PT series for the direct contribution is not fast though.

The uncertainty of the c-quark contribution is straightforward to estimate in view of explicit formulas. The main error comes from the uncertainty of the c-quark mass. In the next-to-leading order one has from eq. (25)

$$\delta\Delta^c(M_\tau^2) = -\frac{4}{3}(1 + a_s)\frac{2\delta m_c}{m_c} = -\frac{8}{3}(1 + a_s)\frac{\delta m_c}{m_c} = \pm 0.330 \qquad (36)$$

for $m_c = M_\tau \pm 0.2$ GeV and $a_s = 0.1$. This is a very large uncertainty. The contribution $\Delta^c(M_\tau^2)$ in eq. (35) is small because the c quark almost decouples but the uncertainty of $\Delta^c(M_\tau^2)$ is large. The uncertainty is mainly given by the variation of $\ln(M_\tau^2/m_c^2)$ in eq. (25) and is independent of an absolute value of the contribution. For the central value $m_c = M_\tau$ one would find a vanishing contribution in the leading order but its uncertainty would almost stay unchanged and equal to 0.330. Also the c-quark mass is not very large and the convergence of the PT expansion in eqs. (25,35) is slow.

Note that for estimating the c-quark contribution we do not take into account the a_s uncertainty. The reason is that uncertainties of the quark mass and of a_s are strongly correlated. Indeed, to the leading order one can find from eq. (25)

$$\delta\Pi^{heavy}(M_\tau^2, 0) = \frac{15}{4}\delta a_s \qquad (37)$$

for independent variation of a_s. However, eq. (25) can be rewritten in terms of the running mass $\bar{m}_c(\mu^2)$. To the first order in a_s the relation between masses reads

$$m_c = \bar{m}_c(\mu^2) \left\{ 1 + a_s(\mu^2) \left(\ln \frac{\mu^2}{\bar{m}_c^2(\mu^2)} + \frac{4}{3} \right) \right\}$$ (38)

that leads to the change in eq. (25)

$$\ln \frac{\mu^2}{m_c^2} + a_s \left(\frac{15}{4} + \ln \frac{\mu^2}{m_c^2} \right) \quad \rightarrow \quad \ln \frac{\mu^2}{\bar{m}_c^2(\mu^2)} + a_s \left(\frac{13}{12} - \ln \frac{\mu^2}{\bar{m}_c^2(\mu^2)} \right).$$

The NLO result for the polarization function in terms of the running mass now reads

$$\Pi_{runmass}^{heavy}(\mu^2, 0) = \ln \frac{\mu^2}{\bar{m}_c^2(\mu^2)} + a_s \left(\frac{13}{12} - \ln \frac{\mu^2}{\bar{m}_c^2(\mu^2)} \right).$$

This result leads to the uncertainty

$$\delta\Pi_{runmass}^{heavy}(M_\tau^2, 0) = \left(\frac{13}{12} - \ln \frac{M_\tau^2}{\bar{m}_c^2(\mu^2)} \right) \delta a_s$$

which is by factor 4 smaller numerically than the previous result eq. (37). The rest of the uncertainty is now in the relation between the pole and running mass given in eq. (38) that represents a regular change of variables in the finite order PT and is under rather a strict control. We work with the pole mass and assume that the uncertainty of the polarization function at the origin is saturated by the uncertainty of the pole mass. It is also assumed that the uncertainty of the pole mass is estimated such that it includes an uncertainty of a_s.

The total finite renormalization between the fine structure constant and the $\overline{\text{MS}}$-scheme EM coupling at M_τ is given by

$$\Delta^{(4)}(M_\tau^2) = \Delta^{lept}(M_\tau^2) + \Delta^{uds}(M_\tau^2) + \Delta^c(M_\tau^2)$$

$$= 22.0109 + 9.9312 + 0.84679 = 32.7889$$ (39)

that leads to

$$\frac{3\pi}{\bar{\alpha}^{(4)}(M_\tau^2)} = \frac{3\pi}{\alpha} - \Delta^{(4)}(M_\tau^2) = \frac{3\pi}{\alpha} - 32.7889.$$ (40)

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The low energy normalization value for the EM coupling in the $\overline{\text{MS}}$ scheme reads

$$\frac{1}{\bar{\alpha}^{(4)}(M_\tau^2)} = 133.557 \,. \tag{41}$$

We now consider the uncertainty of this central result. The lepton contributions are treated as exact so the number from eq. (27) has no errors. The errors due to light quarks are given in eq. (30). Note that one could reduce the sensitivity of $\Delta^{uds}(M_\tau^2)$ to a_s, the error of which dominates the total error in eq. (30), by taking F_R from experiment through the leptonic decay width of the ρ meson (and of the ω and φ mesons in other light quark channels). Then the resonance contribution F_R/m_R^2 does not depend on a_s. The first duality relation fixes s_0 immediately using the fact that power corrections of order $1/q^2$ are absent in the OPE. However, this procedure introduces an experimental error due to an uncertainty in the ρ-meson leptonic decay width

$$\Gamma_{ee}^\rho = 6.77 \pm 0.32 \text{ keV} \,.$$

This uncertainty leads to almost the same error for the final quantity $\Delta^{uds}(M_\tau^2)$ as the uncertainty in a_s. It seems to be natural. Indeed, the strong coupling at low energies is extracted from the τ data in which the ρ-meson contribution constitutes an essential part. This example shows how the coupling constant encodes information on the experimental data. Another point about using Γ_{ee}^ρ for the lowest resonance contribution is that the consistency with OPE is less strict for such a procedure (no dimension-four operators participate). Still having in mind the possibility of further improvement through experiment we consider our estimate of the error given in eq. (30) as rather conservative.

The uncertainty of the c-quark contribution is given in eq. (36). Collecting all together one finds the final prediction

$$\Delta^{(4)}(M_\tau^2) = 32.7889 \pm 0.078_{light} \pm 0.330_c \tag{42}$$

and

$$\frac{3\pi}{\bar{\alpha}^{(4)}(M_\tau^2)} = \frac{3\pi}{\alpha} - \Delta^{(4)}(M_\tau^2) = \frac{3\pi}{\alpha} - (32.7889 \pm 0.078_{light} \pm 0.330_c). \qquad (43)$$

Eq. (43) is the main result for the low energy normalization of the running EM coupling. For the coupling itself it reads

$$\frac{1}{\bar{\alpha}^{(4)}(M_\tau^2)} = 133.557 \pm 0.0083_{light} \pm 0.0350_c \qquad (44)$$

and

$$\bar{\alpha}^{(4)}(M_\tau^2) = 1.0261\alpha.$$

This value $\bar{\alpha}^{(4)}(M_\tau^2)$ (or equivalently $\Delta^{(4)}(M_\tau^2)$) represents the boundary (initial) condition for the running. With this value known the EM coupling can be run to other scales. The final goal is $M_Z = 91.187$ GeV where high precision tests of SM are done. As will be seen later the running itself is very precise numerically and the main uncertainty in the running EM coupling at larger scales is due to the boundary condition eq. (44). The boundary condition eq. (44) has rather a big uncertainty because of the error of the c-quark mass mainly. The uncertainty due to the light quark contribution is reasonably small. It is dominated by the error in $a_s(M_\tau)$ which is mainly theoretical, i.e. related to the truncation of PT series used for describing the τ-lepton decay data. The uncertainty in $a_s(M_\tau)$ can be reduced if some other sources for its determination are used in addition to the τ system. Reducing the c-quark pole mass uncertainty requires more accurate treatment of the threshold region of $c\bar{c}$ production which is rather a challenging problem in QCD.

5   RG evolution in MS scheme

With the boundary value known at sufficiently large scale one can perturbatively run the EM coupling to larger scales. The final goal is the determination of the numerical

value for the EM coupling at M_Z where high precision tests of the standard model are performed. The running itself (as a functional) is extremely precise because β-functions are very well known. The precision of running is affected by the initial value of a_s which is chosen to be $a_s^{(3)}(M_\tau^2)$ and by the b-quark mass m_b. We discuss them in detail later.

5.1 Basic relations for RG evolution

For the evolution between the τ-lepton mass $M_\tau = 1.777$ GeV (numerically $m_c = M_\tau$) and $M_Z = 91.187$ GeV the number of active quarks is either 4 or 5 and only one threshold at m_b is encountered. The evolution equation (running) is written in the form

$$\mu^2 \frac{d}{d\mu^2}\left(\frac{3\pi}{\bar{\alpha}(\mu^2)}\right) = 3\left(1 + \frac{3}{4}\frac{\bar{\alpha}}{\pi}\right) + \left(\frac{10}{3} + \frac{1}{3}\theta_b\right) + \left(\frac{17}{18} + \frac{1}{36}\theta_b\right)\frac{\bar{\alpha}}{\pi}$$
$$- \left(\frac{34}{27} + \frac{1}{27}\theta_b\right)\frac{\bar{\alpha}}{4\pi}a_s + a_s h^{\mathrm{QCD}}(a_s)\,. \tag{45}$$

Here θ_b is a parameter for the b-quark presence, $n_f = 4 + \theta_b$. From M_τ to m_b one has $n_f = 4$ and $\theta_b = 0$ while from m_b to M_Z one has $n_f = 5$ and $\theta_b = 1$. In eq. (45) the strong coupling $a_s(\mu^2)$ obeys RGE

$$\mu^2 \frac{d}{d\mu^2}a_s(\mu^2) = \beta(a_s(\mu^2)) + a_s^2 \frac{\bar{\alpha}}{8\pi}\left(\sum_q e_q^2\right) \tag{46}$$

with

$$\beta(a_s) = -a_s^2(\beta_0 + \beta_1 a_s + \beta_2 a_s^2 + \beta_3 a_s^3) + \mathcal{O}(a_s^6) \tag{47}$$

being the strong interaction β-function. In QCD one has

$$h^{\mathrm{QCD}}(a_s) = \left(\frac{10}{3} + \frac{1}{3}\theta_b\right)\left\{1 + a_s\left(\frac{287}{144} - \frac{11}{72}\theta_b\right)\right.$$
$$+a_s^2\left(\frac{38551}{15552} - \frac{7595}{7776}\theta_b - \frac{77}{3888}\theta_b^2 - \frac{55}{54}\zeta(3)(1 + \theta_b)\right)\right\}$$
$$+ a_s^2\left(\frac{2}{3} - \frac{1}{3}\theta_b\right)^2\left(\frac{55}{72} - \frac{5}{3}\zeta(3)\right) \tag{48}$$

where the first two lines give the 'direct' contribution and the third line gives light-by-light contribution which is written separately because of its different color structure. This result is obtained from the photon renormalization constant given in [25] and explicitly written in [41, 48]. It was used in ref. [23] for calculation of the evolution of the EM coupling constant. Numerically, one finds

$$
h^{\mathrm{QCD}}(a_s) = \left(\frac{10}{3} + \frac{1}{3}\theta_b\right)\left(1 + a_s(1.993 - 0.153\theta_b) + a_s^2(1.26 - 2.20\theta_b - 0.02\theta_b^2)\right)
$$
$$
+ a_s^2(-0.55 + 0.55\theta_b - 0.14\theta_b^2). \tag{49}
$$

Coefficients of the EM β-function in eqs. (45,48,49) are very small that makes the convergence of PT series for the evolution very fast. Eqs. (45,46) should be solved simultaneously. However, the EM coupling $\bar{\alpha}(\mu)$ is small, therefore, we neglect its running in corrections and substitute there the value numerically equal to the fine structure constant α. Then one has to integrate the trajectory of the strong coupling $a_s(\mu)$ which is given by the solution to RGE (46). The α correction in the strong coupling β-function is numerically of order a_s^2 and formally should be retained if a_s^4 terms in the β-function are retained. However, the main contribution to the running is given by the partonic part of the EM β-function in eq. (45), i.e. independent of both EM and strong couplings. Other terms give only small corrections. As for practical calculations, one can do everything numerically, however, it happens that the two-loop running gives almost the same result as the exact treatment. With the two-loop accuracy the integration can be done analytically in a simple form. Indeed, for $\beta(a_s) = -\beta_0 a_s^2 - \beta_1 a_s^3$ one finds

$$
\int_{\mu_1^2}^{\mu_2^2} a_s(\xi)d\ln\xi = \frac{1}{\beta_0}\ln\left(\frac{\beta_0/a_s(\mu_2^2) + \beta_1}{\beta_0/a_s(\mu_1^2) + \beta_1}\right),
$$
$$
\int_{\mu_1^2}^{\mu_2^2} a_s(\xi)^2\frac{d\xi}{\xi} = -\frac{1}{\beta_1}\ln\left(\frac{\beta_0 + \beta_1 a_s(\mu_2^2)}{\beta_0 + \beta_1 a_s(\mu_1^2)}\right) \tag{50}
$$

where the NLO solution for the running coupling $a_s(\mu)$ is given by

$$\ln\left(\frac{\mu^2}{\Lambda^2}\right) = \Phi(a_s) = \int^{a_s} \frac{d\xi}{-\xi^2(\beta_0 + \beta_1\xi)} = \frac{1}{a_s\beta_0} + \frac{\beta_1}{\beta_0^2}\ln\left(\frac{a_s\beta_0^2}{\beta_0 + a_s\beta_1}\right). \quad (51)$$

In NNLO it is also possible to perform integration explicitly but results are too awkward to present here. In fact, the NLO integration as given in eqs. (50,51) is rather precise numerically and can be used for preliminary estimates. We, however, avoid any approximation of this sort (cf ref. [23]) and give numbers for a direct numerical treatment of RG equations (45,46) with the four-loop strong coupling β-function from eq. (47) and $h^{\rm QCD}$-function from eq. (48).

The solution to RGE can be used for the range of μ where the corresponding effective theory (with a given number of active leptons and quarks) is valid. Because decoupling is not automatic one should explicitly take into account thresholds.

5.2 Running to m_b

The first scale of interest is m_b where there is an important physics due to $b\bar{b}$ production near threshold and accurate data on Υ resonances (note that the real threshold energy is, in fact, $2m_b$ but the matching is defined at m_b). We use $m_b = 4.8\pm0.2$ GeV as determined in ref. [49].

In the approximation when the EM coupling is taken to be constant in the correction, the contribution of leptons is given by

$$\Delta_{\tau b}^{lept}(m_b^2) = 3(1 + \frac{3}{4}\frac{\alpha}{\pi})\ln\frac{m_b^2}{M_\tau^2} = 3(1 + \frac{3}{4}\frac{\alpha}{\pi})\cdot 1.98738 = 5.9725. \quad (52)$$

The hadronic part is more involved. In the energy range from M_τ to m_b the number of active quarks is 4 or $\theta_b = 0$. The partonic part of quark contribution reads

$$\Delta_{\tau b}^{(0)}(m_b^2) = N_c\sum_q e_q^2\left(1 + e_q^2\frac{3}{4}\frac{\bar{\alpha}}{\pi}\right)\ln\frac{m_b^2}{M_\tau^2} = \left(\frac{10}{3} + \frac{17}{18}\frac{\alpha}{\pi}\right)\ln\frac{m_b^2}{M_\tau^2} = 6.62896 \quad (53)$$

where we use $\bar{\alpha} = \alpha$. The result is independent of strong coupling constant (the parton model result without real QCD interaction). The quark part beyond the partonic approximation requires integration of the evolution trajectory of the strong coupling in $n_f = 4$ effective theory. The initial value of the strong coupling is $a_s^{(4)}(M_\tau^2) = 0.102001$ as was obtained from matching at M_τ^2 for the c-quark contribution. In NLO one finds still a sizable contribution

$$\Delta_{\tau b}^{(1)}(m_b^2) = (\frac{10}{3} - \frac{17}{54}\frac{\alpha}{\pi})I_{\tau b}^{(1)} = 0.54878 \,. \tag{54}$$

The NNLO contribution proportional to a_s^2 in eq. (45)

$$\Delta_{\tau b}^{(2)}(m_b^2) = \frac{10}{3}\frac{287}{144}I_{\tau b}^{(2)} = 0.091848 \tag{55}$$

and the NNNLO contribution proportional to a_s^3 in eq. (45)

$$\Delta_{\tau b}^{(2)}(m_b^2) = \left(\frac{200675}{23328} - \frac{335}{81}\zeta(3)\right)I_{\tau b}^{(3)} = 3.63085 I_{cb}^{(3)} = 0.0042699 \tag{56}$$

give only small corrections. Here

$$I_{\tau b}^{(n)} = \int_{M_\tau^2}^{m_b^2}(a_s^{(4)}(s))^n \frac{ds}{s} \,.$$

A total correction to the parton model result (i.e. QCD contribution)

$$\Delta_{\tau b}^{(hadcor)}(m_b^2) = \Delta_{\tau b}^{(1)}(m_b^2) + \Delta_{\tau b}^{(2)}(m_b^2) + \Delta_{\tau b}^{(3)}(m_b^2) = 0.644899 \tag{57}$$

is much smaller than the leading partonic result $\Delta_{\tau b}^{(0)}(m_b^2)$. For the EM coupling at m_b one finds

$$\begin{aligned}\frac{3\pi}{\bar{\alpha}^{(4)}(m_b^2)} &= \frac{3\pi}{\bar{\alpha}^{(4)}(M_\tau^2)} - (5.9725 + 6.62896 + 0.644899) \\ &= \frac{3\pi}{\bar{\alpha}^{(4)}(M_\tau^2)} - 13.2464 \,. \end{aligned} \tag{58}$$

Lepton and parton contributions dominate. Collecting all together one finds

$$\frac{3\pi}{\bar{\alpha}^{(4)}(m_b^2)} = \frac{3\pi}{\bar{\alpha}^{(4)}(M_\tau^2)} - 13.2464$$

$$= \frac{3\pi}{\alpha} - (32.7889 + 13.2464) = \frac{3\pi}{\alpha} - 46.0353 \,. \tag{59}$$

And finally

$$\frac{1}{\bar{\alpha}^{(4)}(m_b^2)} = 132.152 \tag{60}$$

or

$$\bar{\alpha}^{(4)}(m_b^2) = 1.037\alpha \,. \tag{61}$$

This number can be used for the Υ-resonance physics analysis.

Because decoupling is not explicit in mass-independent renormalization schemes there is another EM coupling parameter related to the scale m_b. Upon changing the number of active quarks to $n_f = 5$ one obtains

$$\frac{3\pi}{\bar{\alpha}^{(4)}(m_b^2)} = \frac{3\pi}{\bar{\alpha}^{(5)}(m_b^2)} + \Pi^{bfull}(\mu^2 = m_b^2, 0) \,. \tag{62}$$

The polarization function $\Pi^{bfull}(\mu^2 = m_b^2, 0)$ which gives the corresponding shift for the EM constant is written in terms of the effective strong coupling constant $a_s^{(5)}(m_b^2)$. A numerical value for $a_s^{(5)}(m_b^2)$ is obtained through matching the strong coupling at the scale m_b. The running of the coupling $a_s^{(4)}(M_\tau^2) = 0.102$ to $m_b = 4.8$ GeV gives $a_s^{(4)}(m_b^2) = 0.06851$ ($\alpha_s^{(4)}(m_b^2) = 0.2152$). Then matching at m_b results in $a_s^{(5)}(m_b^2) = 0.06869$. With this number the result of matching for the EM constant is

$$\Delta^b(m_b^2) = \Pi^{bfull}(\mu^2 = m_b^2, 0) = \Pi^{bdir}(\mu^2 = m_b^2, 0) + \Pi^{bloop}(\mu^2 = m_b^2, 0)$$

$$= 0.00024_{EM} + 0.00358_{loop} + 0.08587 + 0.03437 = 0.1241 \,. \tag{63}$$

The EM contribution is totally negligible. The loop contribution is rather small. The PT convergence of the direct contribution is not fast and is similar to the c-quark case. One has

$$\Delta^b(m_b^2) = 0.1241 \,. \tag{64}$$

Finally, the EM couplings of $n_f = 4$ and $n_f = 5$ effective theories in the vicinity of m_b are related by

$$\frac{3\pi}{\bar{\alpha}^{(5)}(m_b^2)} = \frac{3\pi}{\bar{\alpha}^{(4)}(m_b^2)} - \Delta^b(m_b^2) = \frac{3\pi}{\bar{\alpha}^{(4)}(m_b^2)} - 0.1241\,. \tag{65}$$

Explicitly one finds

$$\bar{\alpha}^{(5)}(m_b^2) = \frac{1}{132.138} = 1.0001\bar{\alpha}^{(4)}(m_b^2)\,.$$

This difference can be safely neglected in applications for Υ-resonance physics.

The uncertainty due to m_b is tiny. Indeed, the error in the b-quark mass leads to the uncertainty

$$\delta\Delta^b(m_b^2) = -\frac{1}{3}(1 + a_s^{(5)}(m_b^2))\frac{2\delta m_b}{m_b} = -\frac{2}{3}(1 + a_s^{(5)}(m_b^2))\frac{\delta m_b}{m_b} = \pm 0.030\,. \tag{66}$$

There are two reasons for such a smallness in comparison to the c-quark case: the electric charge of b quark $|e_b|$ is two times smaller than $|e_c|$ and the relative uncertainty of the b-quark mass m_b ($\delta m_b/m_b$) is much smaller than that of the c-quark mass. Note that because the contribution $\Delta^b(m_b^2)$ itself is small the relative uncertainty $\delta\Delta^b(m_b^2)/\Delta^b(m_b^2)$ is huge. However, one cannot use it here. Even for $\Delta^b(m_b^2) = 0$ the uncertainty $\delta\Delta^b(m_b^2)$ is basically 0.030.

5.3   Running from m_b to M_Z.

In this subsection we describe the evolution of the EM coupling constant $\bar{\alpha}^{(5)}(m_b^2)$ from $m_b = 4.8$ GeV to $M_Z = 91.187$ GeV. Various contributions according to eqs. (45,48) are:

a) Leptonic contribution

$$\Delta_{bZ}^{lept} = 3\left(1 + \frac{3}{4}\frac{\alpha}{\pi}\right)\ln\frac{M_Z^2}{m_b^2} = 17.6966\,; \tag{67}$$

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b) Leading quark partonic, a_s-independent, contribution

$$\Delta_{bZ}^{(0)} = \left(\frac{11}{3} + \frac{35}{36}\frac{\alpha}{\pi}\right) \ln \frac{M_Z^2}{m_b^2} = 21.6048 ; \tag{68}$$

c) the NLO contribution with $a_s^{(5)}(m_b^2) = 0.068694$ as initial value for the evolution trajectory

$$\Delta_{bZ}^{(1)} = \left(\frac{11}{3} - \frac{35}{108}\frac{\alpha}{\pi}\right) I_{bZ}^{(1)} = 1.0780 ; \tag{69}$$

d) the NNLO contribution proportional to a_s^2

$$\Delta_{bZ}^{(2)} = \frac{11}{3}\frac{265}{144} I_{bZ}^{(2)} = 0.10213 ; \tag{70}$$

e) the NNNLO contribution proportional to a_s^3

$$\Delta_{bZ}^{(3)} = \left(\frac{257543}{46656} - \frac{620}{81}\zeta(3)\right) I_{bZ}^{(3)} = -3.68089 I_{bZ}^{(3)} = -0.002954 . \tag{71}$$

Here

$$I_{bZ}^{(n)} = \int_{m_b^2}^{M_Z^2} (a_s^{(5)}(s))^n \frac{ds}{s} .$$

A total QCD correction to the partonic result

$$\Delta_{bZ}^{(hadcor)} = \Delta_{bZ}^{(1)} + \Delta_{bZ}^{(2)} + \Delta_{bZ}^{(3)} = 1.1772 \tag{72}$$

is small compared to the leading quark partonic, a_s-independent, contribution given in eq. (68). The total effect of running on the interval from m_b to M_Z

$$\Delta_{bZ}^{lept} + \Delta_{bZ}^{(0)} + \Delta_{bZ}^{(hadcor)} = 17.697 + 21.6048 + 1.1772 = 40.479 \tag{73}$$

is dominated by leptons and by the quark partonic contribution. We find the EM coupling at M_Z expressed through the EM coupling at m_b in the form

$$\frac{3\pi}{\bar{\alpha}^{(5)}(M_Z^2)} = \frac{3\pi}{\bar{\alpha}^{(5)}(m_b^2)} - 40.479 . \tag{74}$$

This equation gives the relation between the running EM couplings necessary for applications in b- and Z-physics.

Collecting together eqs. (40,58,64,74) we find the absolute value of the running EM coupling at M_Z expressed through the fine structure constant α

$$\frac{3\pi}{\bar{\alpha}^{(5)}(M_Z^2)} = \frac{3\pi}{\alpha} - 32.7889(match\ M_\tau) - 13.2464(run\ m_c 2m_b)$$

$$- 0.1241(match\ m_b) - 40.479(run\ m_b 2M_Z) = \frac{3\pi}{\alpha} - 86.6384 \qquad (75)$$

and

$$\frac{1}{\bar{\alpha}^{(5)}(M_Z^2)} = \frac{1}{\alpha} - 86.6384/(3\pi) = 137.036 - 9.1926 = 127.843 \,. \qquad (76)$$

The result can be written as a relation between the running EM coupling and the fine structure constant

$$\bar{\alpha}^{(5)}(M_Z^2) = 1.0719\alpha \,. \qquad (77)$$

This number can be used for the Z-boson peak analysis.

6   Summary of results

In this section we give a brief summary of the calculation paying attention to uncertainties of the results.

The uncertainty for the low energy normalization value at the τ mass is given in eqs. (42,43,44). It is largely dominated by the uncertainty due to the c-quark contribution. Adding uncertainties

$$\delta\Delta^{(4)}(M_\tau^2) = \pm 0.078_{light} \pm 0.330_c \,.$$

in quadrature one finds

$$\delta\Delta^{(4)}(M_\tau^2) = \pm 0.339$$

and

$$0.339/(3\pi) = 0.036.$$

Range	$(0 - M_\tau)$	$M_\tau - m_b$	$m_b - M_Z$	$total$
Δ^{lept}	22.011	5.973	17.697	45.681

Table 1: Leptonic contributions.

Matching	$\Delta^{uds}(M_\tau)$	$\Delta^c(M_\tau)$	$\Delta^b(m_b)$
Value	9.9312 ± 0.078	0.8468 ± 0.330	0.1241 ± 0.030

Table 2: Matching at different scales.

Finally one obtains for the low energy normalization value the result of the form

$$\frac{1}{\bar{\alpha}^{(4)}(M_\tau^2)} = 133.557 \pm 0.036 \,. \tag{78}$$

The total error is dominated by the uncertainty due to the c-quark matching contribution which is mainly given by the uncertainty of the c-quark mass.

For the scales m_b and M_Z, the errors due to running, which are basically because of the uncertainty of the coupling constant a_s, should be included. The running itself is precise because β-functions in eqs. (45,46) are computed up to high order of PT and the coupling constant a_s is rather well known. The dominant contribution comes from leptons and partonic quarks and is independent of the genuine QCD interaction (see Tables 1,3). The EM terms give a tiny correction. In Table 4 the quantity $\int_{M_\tau^2}^{m_b^2} a_s(s)\frac{ds}{s}$ with running for $a_s(s)$ in different orders and with or without EM contribution to the strong β-function is presented. The inclusion of EM terms slows down the decrease of the strong coupling and the integrals are slightly larger; still it is completely negligible numerically. In the leading order we have the uncertainty in integrals due to errors of the initial value of the strong coupling

$$\delta I_{ab}^{(1)} = \frac{L_{ab}}{1 + a_s \beta_0 L_{ab}} \delta a_s \tag{79}$$

Power	$M_\tau - m_b$	$m_b - M_Z$	total
$\Delta^{(0)} \sim a_s^0$	6.6290	21.605	28.234
$\Delta^{(1)} \sim a_s^1$	0.5488	1.078	1.627
$\Delta^{(2)} \sim a_s^2$	0.0918	0.102	0.194
$\Delta^{(3)} \sim a_s^3$	0.0043	-0.003	0.001
total sum	7.2739	22.782	30.056
$\Delta^{(hadcor)}$	0.6449	1.177	1.822

Table 3: Running of powers of a_s.

with $L_{ab} = \ln \mu_b^2 / \mu_a^2$. This equation suffices for error estimates of the QCD contribution into running. NNLO and NNNLO give only small corrections. One can find uncertainty of the running by varying the initial value of a_s. The numerical results are close to the estimate given in eq. (79). Eq. (79), however, has an advantage of being analytical and simple that makes the error evaluation more transparent.

At m_b (and M_Z) errors due to running and due to matching the light quark contribution at M_τ are not independent: both are determined mainly by the uncertainty in a_s. Therefore these errors should be added linearly.

For the interval from M_τ to m_b one has from eq. (79)

$$\delta \Delta_{\tau b}^{hadcor}|_{a_s} = 4.56 \delta a_s = 0.025$$

and the total error (with linearly added errors for matching the light quark contribution at M_τ and running) is

$$\delta \Delta^{(5)}(m_b^2) = \pm (0.078 + 0.025)_{light+run} \pm 0.330_c \pm 0.030_b$$

$$= \pm 0.103_{light+run} \pm 0.330_c \pm 0.030_b \, .$$

Order	$I_{\tau b}^{(1)}$ *with* EM	$I_{\tau b}^{(1)}$ *without* EM
LO	0.169104	0.169100
NLO	0.165542	0.165538
NNLO	0.164938	0.164934
NNNLO	0.164671	0.164667

Table 4: The quantity $I_{\tau b}^{(1)}$ from eq. (54) in different orders of strong β-function and with or without the EM contribution to the strong β-function.

Adding independent errors in quadrature one has

$$\delta\Delta^{(5)}(m_b^2) = \pm 0.347$$

and

$$0.347/(3\pi) = 0.0368 \,.$$

Finally, one finds the uncertainty for the EM coupling at m_b

$$\frac{1}{\bar{\alpha}^{(4)}(m_b)} \approx \frac{1}{\bar{\alpha}^{(5)}(m_b)} = \frac{1}{\alpha} - \Delta^{(5)}(m_b^2)$$

$$= 137.036 - (4.89766 \pm 0.0368) = 132.138 \pm 0.0368 \,. \tag{80}$$

For the scale M_Z the error due to running is estimated by

$$\delta\Delta_{\tau Z}^{hadcor}|_{a_s} = \frac{11}{3}\delta I_{\tau Z}^{(1)}(a_s(m_b)) = 13.6\delta a_s = 0.074$$

which leads to

$$\delta\Delta^{(5)}(M_Z^2) = \pm 0.078_{light} \pm 0.330_c \pm 0.030_b \pm 0.074_{run}$$

$$= \pm 0.152_{light+run} \pm 0.330_c \pm 0.030_b \,. \tag{81}$$

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Adding independent errors in quadrature one has

$$\delta\Delta^{(5)}(M_Z^2) = \pm 0.3646$$

and

$$0.3646/(3\pi) = 0.03869$$

These estimates give the error for the coupling at M_Z

$$\frac{1}{\bar{\alpha}^{(5)}(M_Z^2)} = 127.843 \pm 0.039\,. \tag{82}$$

Eq. (82) is a final result. However, it cannot be directly compared with the results of the standard analyses because the quantity in eq. (82) is defined in a different scheme. We consider the uncertainty in eq. (82) (the part of which related to the running is estimated analytically for transparency) as rather conservative.

7   Comparison with other schemes

With the number from eq. (82) one can find the on-shell parameter α_{os} at M_Z used in the literature. Indeed, because of the relation

$$\frac{3\pi}{\alpha_{os}(\mathbf{q^2})} = \frac{3\pi}{\alpha} + \Pi_{os}(\mathbf{q^2}) = \frac{3\pi}{\bar{\alpha}^{(5)}(\mu^2)} + \Pi^{(5)}(\mu^2, \mathbf{q^2})\,. \tag{83}$$

one has to compute $\Pi^{(5)}(\mu^2, \mathbf{q^2})$ in $n_f = 5$ effective theory at the point $q^2 \sim M_Z^2$. Note that we have restored a notation q^2 in addition to the usual quantity $\mathbf{q^2}$: the new variable q^2 will be used in Minkowskian domain. For computing the leading part of $\Pi^{(5)}(\mu^2, \mathbf{q^2})$ in the kinematical range $\mu^2 \sim \mathbf{q^2} \sim M_Z^2$ one can consider all five active quarks (u, d, s, c, b) and all three leptons as massless and use eq. (10) with the only change because of a different number of active quarks which is now 5 instead of 3. This change affects only $\mathcal{O}(a_s^2)$ order in eq. (10) and changes nothing for leptons in

the NLO approximation. One has for the generic light polarization function of quarks

$$\Pi^{light-n_f}(\mu^2, \mathbf{q}^2) = \ln\frac{\mu^2}{\mathbf{q}^2} + \frac{5}{3} + a_s\left(\ln\frac{\mu^2}{\mathbf{q}^2} + \frac{55}{12} - 4\zeta(3)\right)$$

$$+a_s^2\left\{\frac{\beta_0(n_f)}{2}\ln^2\frac{\mu^2}{\mathbf{q}^2} + \left(\frac{365}{24} - \frac{11}{12}n_f - 4\beta_0(n_f)\zeta(3)\right)\ln\frac{\mu^2}{\mathbf{q}^2}\right.$$

$$\left.+\frac{41927}{864} - \frac{3701}{1296}n_f - \left(\frac{829}{18} - \frac{19}{9}n_f\right)\zeta(3) + \frac{25}{3}\zeta(5)\right\} \tag{84}$$

with $\beta_0(n_f) = (11 - 2/3n_f)/4$. For a more accurate evaluation of $\Pi^{(5)}(\mu^2, \mathbf{q}^2)$ at the scale M_Z we retain the leading corrections due to the b-, c-quark and τ-lepton masses and the leading correction due to the top quark contribution. One finds

$$\Pi^{(5)}(\mu^2, \mathbf{q}^2) = 3\Pi^{light-lept}(\mu^2, \mathbf{q}^2) + \frac{11}{3}\Pi^{light-quark}(\mu^2, \mathbf{q}^2)$$

$$-\frac{1}{3}\frac{6m_b^2}{\mathbf{q}^2} - \frac{4}{3}\frac{6m_c^2}{\mathbf{q}^2} - \frac{6M_\tau^2}{\mathbf{q}^2} + \Delta_{(t)}\Pi^{(5)}(\mu^2, \mathbf{q}^2). \tag{85}$$

Note that the power correction due to a quark mass is exact up to $\mathcal{O}(a_s^2)$ order when expressed through the pole mass. The corrections due to the top quark contribution for the quantity $\Pi^{(5)}(\mu^2, q^2)$ at $\mathbf{q}^2 \approx M_Z^2$ can be computed as a power series in $\mathbf{q}^2/m_t^2$; the expansion parameter $\mathbf{q}^2/m_t^2$ is small at the point $\mathbf{q}^2 = M_Z^2$ for $m_t = 175$ GeV. Indeed, retaining only the leading term and first corrections one has

$$\Delta_{(t)}\Pi^{(5)}(M_Z^2, \mathbf{q}^2) = -\frac{4}{15}\frac{\mathbf{q}^2}{m_t^2}\left\{1 + \frac{410}{81}a_s^{(5)}(M_Z^2) - \frac{3}{28}\frac{\mathbf{q}^2}{m_t^2}\right\}. \tag{86}$$

A typical expansion in eq. (86) reads

$$\Delta_{(t)}\Pi^{(5)}(M_Z^2, M_Z^2) = -0.0724 - 0.0138_{a_s} + 0.0021_{M_Z} = -0.0841 \tag{87}$$

with obvious notation indicating the origin of different contributions. We do not take into account bosons therefore the W-boson loops should be analyzed separately.

To calculate the on-shell coupling $\alpha_{os}(\mathbf{q}^2)$ at the scale M_Z using eq. (83) one can use either $\mathbf{q}^2 = M_Z^2$ (Euclidean definition) [50] or $q^2 = -\mathbf{q}^2 = M_Z^2$ with taking the real

part of the correlator (Minkowskian definition). The Minkowskian definition is usually discussed in the literature. Note that we calculate not the e^+e^- scattering amplitude at the total energy M_Z ($q^2 = M_Z^2$) which definitely should be taken at a physical point on the cut in the case of cross section calculations, but the coupling constant which parameterizes this amplitude at the scale M_Z. Within the RG approach the scale of the parameters of an effective theory valid in a given energy range should not coincide with any actual physical value of the energy or momentum squared (see e.g. [51]).

First we use a Euclidean definition for the on-shell coupling which is consistently perturbative and requires computation of the correlator $\Pi^{(5)}(M_Z^2, \mathbf{q}^2)$ in a deep Euclidean domain for $\mathbf{q}^2 = M_Z^2$. Using eqs. (85,86) one finds an expansion

$$\Pi^{(5)}(M_Z^2, M_Z^2) = 11.1111 - 0.03097_{a_s} + 0.00112_{a_s^2}$$

$$-0.00168_{EM} - 0.00554_b - 0.00304_c - 0.00228_\tau - 0.0841_t$$

$$= 11.0796 - 0.01086_{bc\tau} - 0.0841_t \, . \tag{88}$$

Note that the EM correction is numerically of the order a_s^2. Still these corrections are very small. From eq. (83) we find

$$\frac{3\pi}{\alpha_{os}(M_Z^2)} = \frac{3\pi}{\bar{\alpha}^{(5)}(M_Z^2)} + \Pi^{(5)}(M_Z^2, M_Z^2)$$

$$= \frac{3\pi}{\alpha} - 86.6384 + 11.0796 - 0.01086_{bc\tau} - 0.0841_t$$

$$= -75.5588 - 0.01086_{bc\tau} - 0.0841_t \, .$$

For clarity we retain the contribution of power corrections separately for further comparison with the results in Minkowskian domain. One has numerically

$$\frac{1}{\alpha_{os}(M_Z^2)} = \frac{1}{\alpha} - (75.5588 + 0.01086_{bc\tau} + 0.0841_t)/(3\pi)$$

$$= 137.036 - 8.0271 = 129.009 \, .$$

Because the error estimate in eq. (82) is not affected by this change of scheme (too small and rather precise contributions are added), the final result for the on-shell coupling within Euclidean definition reads

$$\frac{1}{\alpha_{os}(M_Z^2)} = 129.009 \pm 0.039 \,. \tag{89}$$

However, the reference values for the on-shell coupling available in the literature are given in Minkowskian domain for $q^2 = -\mathbf{q}^2 = M_Z^2$, i.e. for the real part of the correlator $\Pi^{(5)}(M_Z^2, q^2)$ computed on the physical cut. Within the approximation used, going to the Minkowskian domain of momenta q^2 changes only the a_s^2 order term and power corrections in eq. (88). Indeed, in eq. (84) the only term which is numerically affected by the transition to the Minkowskian domain is $\ln^2(\mu^2/\mathbf{q}^2)$ with the following change as compared to the Euclidean result

$$\text{Re}\,\left\{\ln^2\left(\frac{\mu^2}{-M_Z^2}\right)\right\} = \ln^2\left(\frac{\mu^2}{M_Z^2}\right) - \pi^2 \,. \tag{90}$$

Instead of eq. (88) one finds

$$\text{Re}\,\left\{\Pi^{(5)}(M_Z^2, -M_Z^2)\right\} = 11.0796$$

$$- 0.04893_{\pi^2} + 0.01086_{bc\tau Mink} + 0.08828_{tMink} = 11.1298 \tag{91}$$

and

$$\text{Re}\,\frac{1}{\alpha_{os}(-M_Z^2)} = 137.036 - (86.6384 - 11.1298)/(3\pi) = 129.024 \,.$$

The final result for the on-shell coupling with Minkowskian definition is

$$\text{Re}\,\frac{1}{\alpha_{os}(-M_Z^2)} = 129.024 \pm 0.039 \,. \tag{92}$$

The difference between the central values for the couplings in eq. (89) which gives the Euclidean definition and in eq. (92) which gives the Minkowskian definition is 0.015. Note that the Euclidean definition was considered in ref. [50] where the numerical

difference about 0.02 from the Minkowskian definition has been found from rather a simplified treatment. It is close to the present more accurate result 0.015. Note that the point M_Z is safe for the PT calculation in Minkowskian domain for the approximation used (no singularities of the spectrum near this point). At other points it is not so even in the approximation we work. For instance, if the Υ-resonance mass m_Υ is taken as a reference scale then the Euclidean definition is equally applicable at this point being still perturbative while the Minkowskian definition faces the problems that the polarization function on the cut is not smooth. A phenomenological approach based on direct integration of data fails because of the fast change of the spectrum at the location of the Υ resonance which makes the integration with principal value prescription for regularizing the singularities ill-defined. A theory based approach within PT fails at the point m_Υ because PT calculations for the correlator near the threshold on the physical cut ($m_\Upsilon \sim 2m_b$) are not reliable. Therefore, the Minkowskian definition is not uniformly applicable at every scale.

The present paper result given in eq. (92) differs from some recent determinations based on the use of experimental data for integration over the low energy region. For the result of ref. [16]

$$\text{Re}\ \frac{1}{\alpha_{os}(-M_Z^2)} = 128.925 \pm 0.056 \tag{93}$$

the number obtained in the present paper and given in eq. (92) almost touches the reference value eq. (93) within 1σ (σ is a standard deviation). The results of some other groups are concentrated around the same central value as in eq. (93) but with essentially smaller errors. For further comparison, we use the result of ref. [17]

$$\text{Re}\ \frac{1}{\alpha_{os}(-M_Z^2)} = 128.927 \pm 0.023\,. \tag{94}$$

The difference between the value from eq. (92) and the reference result in eq. (94) is $129.024 - 128.927 = 0.097$ which constitutes 2-4σ and can be significant. Therefore we discuss the difference in more detail.

The usual parameterization of the fermionic contributions to the on-shell running EM coupling at M_Z reads

$$\text{Re}\ \frac{1}{\alpha_{os}(-M_Z^2)} = \frac{1}{\alpha}\left(1 - \Delta\alpha_{\text{lep}} - \Delta\alpha_{\text{had}}^{(5)} - \Delta\alpha_{\text{top}}\right)$$

(note that

$$\text{Re}\ \left(\frac{1}{\alpha_{os}(-M_Z^2)}\right) \neq \frac{1}{\text{Re}\ \alpha_{os}(-M_Z^2)}$$

though the difference is tiny). The total leptonic contribution to the electromagnetic coupling constant at M_Z given in the last column of Table 1 reads

$$\Delta^{lep}(M_Z^2) = 45.681.$$

The leptonic part of $\Pi^{(5)}(M_Z^2, -M_Z^2)$ reads

$$\text{Re}\ \left\{\Pi^{(5)lep}(M_Z^2, -M_Z^2)\right\} = 3\text{Re}\ \left(\Pi^{light-lept}(M_Z^2, -M_Z^2)\right) + \frac{6M_\tau^2}{M_Z^2}$$

$$= 4.9988 + 0.0023 = 5.0011\,.$$

The leading order contribution is equal to 5 while the EM and τ-lepton mass corrections are small. For the total leptonic contribution to the on-shell coupling in the Minkowskian domain one finds

$$\Delta\alpha_{\text{lep}} = \frac{\alpha}{3\pi}(45.681 - 5.001) = 314.974 \times 10^{-4}$$

which is close to the number of ref. [17]

$$\Delta\alpha_{\text{lep}}|_{\text{ref}} = (314.19 + 0.78) \times 10^{-4} = 314.97 \times 10^{-4}\,.$$

For the top contribution in the Minkowskian domain one finds from eq. (86) (see also eq. (87))

$$\Delta\alpha_{\text{top}} = \frac{\alpha}{3\pi}(-0.0883) = -0.68 \times 10^{-4}$$

while the number of ref. [17] with a more accurate account for the higher order corrections is

$$\Delta\alpha_{\text{top}}|_{\text{ref}} = -0.70 \times 10^{-4}.$$

The difference is small and is neglected. From the numerical value given in eq. (92) the total contribution of fermions into the shift of the EM coupling is determined to be

$$\Delta\alpha_{\text{lep}} + \Delta\alpha_{\text{top}} + \Delta\alpha_{\text{had}}^{(5)} = 1 - \alpha \cdot (129.024 \pm 0.039) = 0.0584664 \pm 0.000285.$$

Taking $\Delta\alpha_{\text{lep}}$ and $\Delta\alpha_{\text{top}}$ as exact quantities (no errors) one finds the following numerical value for $\Delta\alpha_{\text{had}}^{(5)}$

$$\begin{aligned} \Delta\alpha_{\text{had}}^{(5)} &= (0.0584664 \pm 0.000285) - 0.031497 + 0.000068 \\ &= (270.37 \pm 2.85) \times 10^{-4} \end{aligned} \tag{95}$$

while the result of ref. [17] is

$$\Delta\alpha_{\text{had}}^{(5)}|_1 = (277.45 \pm 1.68) \times 10^{-4}, \tag{96}$$

and the number of ref. [16] is

$$\Delta\alpha_{\text{had}}^{(5)}|_2 = (277.6 \pm 4.1) \times 10^{-4}. \tag{97}$$

The difference between the central values given in eq. (95) and eqs. (96,97) is about 2-4σ depending on the numerical value of the error quoted

$$277.45 - 270.37 = 7.08 \approx 2.5 \cdot 2.85 \approx 4.3 \cdot 1.68 \approx 1.7 \cdot 4.1.$$

This difference can be significant. Therefore, we discuss the sensitivity of our prediction (95) (and of eq. (82) from which it is uniquely obtained) to the numerical values of parameters used in the theoretical calculation of the present paper. If $m_s = 0$ then the ρ- and φ-channels should be theoretically degenerate because there is no reason

for the difference. This means that besides vanishing explicit corrections due to m_s^2 in eq. (22) one should identify m_{Rs} with the resonance in the nonstrange channel, i.e. numerically substitute $m_{Rs} = m_\rho$ into the solution for the IR modifying parameters in eqs. (20). With such changes one finds the result for $\Delta^{uds}(M_\tau^2)$ in the form

$$\Delta^{uds}(M_\tau^2)|_{m_s=0} = 10.23 \tag{98}$$

that generates a numerical shift about 0.3 in the value of $\Delta^{uds}(M_\tau^2)$ as compared to the result for nonvanishing strange quark mass in eq. (30). Note that if the direct integration of low energy data is used then the full dependence of the results on m_s is lost. Only the PT high energy tail depends explicitly on m_s but this dependence is weak. The change in $\Delta\alpha_{\text{had}}^{(5)}$ corresponding to the result in eq. (98) is

$$\Delta\alpha_{\text{had}}^{(5)}|_{m_s=0} - \Delta\alpha_{\text{had}}^{(5)} = 0.3\frac{\alpha}{3\pi} = 0.000232 = 2.3 \times 10^{-4}.$$

The use of the numerical value $m_c = 1.6$ GeV for the c-quark mass instead of $m_c = 1.777$ GeV generates the 0.33 shift in the value of quantity $\Delta^c(M_\tau^2)$ that leads to the following change of $\Delta\alpha_{\text{had}}^{(5)}$

$$\Delta\alpha_{\text{had}}^{(5)}|_{m_c=1.6} - \Delta\alpha_{\text{had}}^{(5)} = 0.33\frac{\alpha}{3\pi} = 0.000256 = 2.6 \times 10^{-4}.$$

Note that this change cannot be found if the direct integration of contributions of actual charmonium resonances is performed. The total shift due to such a change in these parameters compared to eq. (95) is

$$\begin{aligned}\Delta\alpha_{\text{had}}^{(5)}|_{m_s=0,m_c=1.6} &= (270.37 + 2.3 + 2.6) \times 10^{-4} \\ &= (275.27 \pm 2.85) \times 10^{-4}. \tag{99}\end{aligned}$$

The change of the numerical value of the strong coupling constant α_s from $\alpha_s(M_\tau^2) = 0.318$ to $\alpha_s(M_\tau^2) = 0.335$ gives a 0.152 shift in $\Delta^{(5)}(M_Z^2)$ (according to our error

estimates in eq. (81)) to end up with

$$\Delta\alpha_{\text{had}}^{(5)}|_{m_s=0,m_c=1.6,\alpha_s=0.335} = (270.37 + 2.3 + 2.6 + 1.2) \times 10^{-4}$$
$$= (276.47 \pm 2.85) \times 10^{-4}. \tag{100}$$

This result agrees with other estimates within 1σ. The set of numerical values for the relevant parameters used in eq. (100) is rather close to the set used for obtaining the value in eq. (95) ($m_s = 130$ MeV, $m_c = 1.777$ GeV, $\alpha_s^{(3)}(M_\tau^2) = 0.318$). The total shift in $\Delta\alpha_{\text{had}}^{(5)}$ for the new set of parameters in eq. (100) is larger than the total error given in eq. (95) because the total error is computed in quadrature and the change of the spectrum due to $m_s = 0$ (which makes all three light quark channels degenerate) has not been included into the total error. To definitely distinguish between the results of eq. (100) and eq. (95) more precise numerical values of parameters are necessary.

Within the present paper approach we use virtually no real data on cross sections but rely on the numerical values of several theoretical parameters which are extracted from such data. These parameters are the strong coupling constant, quark masses, vacuum condensates. It is generally believed that the real data can be properly described with these parameters if theoretical formulas are sufficiently accurate. In the case of computing the hadronic contribution into the photon vacuum polarization function in Euclidean domain a theoretical description is pretty accurate because PT is applicable and very precise – in fact, the PT results in this area are almost the best ones available among all PT calculations. An additional reason for such a high theoretical precision is that for the calculation of $\Delta\alpha_{\text{had}}^{(5)}$ in Euclidean domain one extracts only very general information encoded in the data – just the integral over the entire spectrum with a smooth weight function and no details of the behavior over specific energy regions. This is the situation when global duality, which is exact by definition (hadron and quark descriptions are supposed to be exactly equivalent in principle), is applicable and is under a strict control numerically within PT. However,

our calculation shows that at the present level of precision the result for $\Delta\alpha_{\text{had}}^{(5)}$ is rather sensitive to the numerical values of the parameters m_c and a_s which should be fixed from the data. The uncertainties of these parameters can be reduced both with better data and better theoretical formulas for extracting the numerical values for these parameters from the data (especially m_c) while the theoretical framework for the calculation of the hadronic contribution itself is already very precise. Using the result given in eq. (95) and formulas for radiative corrections to the Weinberg angle from ref. [52] (assuming Minkowskian definition for the on-shell coupling) we find that the central value of the Higgs-boson mass moves from the reference value $M_H = 100$ GeV for $\Delta\alpha_{\text{had}}^{(5)} = 280.0 \times 10^{-4}$ to $M_H = 191$ GeV for the value $\Delta\alpha_{\text{had}}^{(5)} = 270.37 \times 10^{-4}$ found in the present paper.

8   Conclusion

The technique of calculating $\Delta\alpha_{\text{had}}^{(5)}$ within dimensional regularization and minimal subtraction is straightforward in PT. It heavily uses the renormalization group which is the most powerful tool of modern high precision analyses in particle phenomenology. Because PT is not applicable only at low energies one should modify only the IR region of integration for light quarks: a numerical integration of data at energies higher than $2 \div 3$ GeV is equivalent to the theoretical calculation in PT if both data and theory are properly treated. The present calculation uses virtually no explicit scattering data but the values of the lowest resonance masses for the light-quark vector channels. Other experimental information is encoded through the numerical values of the coupling constant, quark masses, some vacuum condensates.

Minkowskian definition of the on-shell coupling constant is deficient and not applicable at some points. Both $\overline{\text{MS}}$ and on-shell coupling constants in the Euclidean domain can be reliable determined with the use of theoretical formulas already es-

tablished in high orders of PT. In view of future high precision tests of SM at M_Z and two-loop calculations for the observables in this region it seems that the parameterization of the theory with the running EM coupling in the $\overline{\text{MS}}$ scheme is most promising.

The main uncertainty of the hadronic contribution into the running EM coupling constant at M_Z comes from the error of the numerical value of the c-quark mass m_c. The uncertainty of a_s is less important. Unfortunately, the c-quark mass is a quantity which is very complicated to study. The reason for that is its numerical value close to the strong interaction scale of the order of ρ-meson or proton mass. Therefore, m_c should be treated exactly in theoretical formulas, almost no simplifying approximation is applicable in the kinematical range of energies of order m_c. The presence of a mass usually makes the loop calculations within PT technically more difficult. Near the $c\bar{c}$ production threshold where the mass is essential and where its numerical value can be reliably extracted from accurate experimental data, the Coulomb interaction is enhanced that requires to take it into account exactly while the c-quark mass is too small for NRQCD to work well. Finally, the nonPT corrections due to vacuum condensates within OPE are essential numerically in this energy range but they are not well known because they are given by the gluonic operators [53]. And though the coefficient functions of the relevant operators up to dimension eight are calculated [54], the numerical values of their vacuum condensates are poorly known. These reasons make the accurate determination of the c-quark mass difficult. The uncertainty related to the contribution of the c quark to the hadronic vacuum polarization is additionally enhanced because of the c-quark large electric charge. For the b quark, for instance, all the above problems are much less severe. Therefore, the c-quark physics plays an essential part in the Higgs search through radiative corrections.

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